Unterlagen
zur Vorlesung

Flugmechanik 1

Prof. Dr.-Ing. Dieter Scholz, MSME
Vorwort

Flugmechanik 1 an der HAW Hamburg beschäftigt sich schwerpunktmäßig mit Flugleistungen (aircraft performance) darüber hinaus wird ein Einstieg in Flugeigenschaftsrechnungen (aircraft stability and control) gegeben.

Die Vorlesung Flugmechanik 1 wird seit SS99 durchgeführt basierend auf dem Flugmechanik-Skript der University of Limerick, Department of Mechanical & Aeronautical Engineering. Der Autor des Skriptes ist Trevor Young. Mr. Trevor Young ist an der University of Limerick verantwortlich für die Organisation der Kurse im Flugzeugbau. Er lehrt die Fächer Flugmechanik und Flugzeugentwurf.

Contents

1 Introduction to Flight Mechanics and the ISA ................................................................. 5
  1.1 Equations for the International Standard Atmosphere ................................................. 7
  1.2 Height Scales and Conversions ................................................................................. 8
  1.3 Aircraft Speed Definitions and Conversions ............................................................ 10
  1.4 Air Temperature ..................................................................................................... 13
  1.5 Rules of Thumb ..................................................................................................... 14

3 Aircraft Drag and Drag Power ....................................................................................... 15

4 Powerplant Performance ............................................................................................... 16
  4.1 Thrust and Efficiency .............................................................................................. 16
  4.2 Propeller Efficiency ................................................................................................ 19

5 Level, Climbing and Descending Flight ......................................................................... 23
  5.1 Climb and Climb Schedules .................................................................................... 23
  5.2 Time to Climb ........................................................................................................ 25
  5.3 Approximate Climb Calculation .............................................................................. 25

6 Stall, Speed Stability, Turning Performance ................................................................. 27

References .................................................................................................................... 28

Appendix
  A Derivations ............................................................................................................ 29
1 Introduction to Flight Mechanics and the ISA

Fig. 1.1: Basic classification of the atmosphere [15] (modified)

---

1 Numbering of references according to lecture notes. Additional references are listed in a separate Chapter in the back of these notes.
Fig. 1.2: Temperature distribution in the standard atmosphere (Anderson 1989)
1.1 Equations for the International Standard Atmosphere

Equations for the troposphere

Troposphere from 0 m = 0 ft to 11000 m = 36089 ft (geopotential height)

\[ T = T_0 - L \cdot H \]

- \( T_0 = 288.15 \) K
- \( L = 0.0065 \) K/m = 6.5 K/km = 1.9812 \( \cdot 10^{-3} \) K/ft

\[ \delta = \frac{p}{p_0} = (1 - k_a \cdot H)^{2.5588} \]

- \( p_0 = 101325 \) Pa = 1013.25 hPa = 1.01325 bar
- \( k_a = 2.2558 \cdot 10^{-5} \) 1/m = 0.022558 1/km = 6.8756 \( \cdot 10^{-6} \) 1/ft

\[ \sigma = \frac{\rho}{\rho_0} = (1 - k_a \cdot H)^{4.2558} \]

- \( \rho_0 = 1.225 \) kg/m³
- \( k_a = 2.2558 \cdot 10^{-5} \) 1/m = 0.022558 1/km = 6.8756 \( \cdot 10^{-6} \) 1/ft

Equations for the Stratosphere

Stratosphere from 11000 m = 36089 ft to 20000 m = 65617 ft (geopotential height)

\( T = T_S = 216.65 \) K = -56.5 °C = const

\[ \frac{\sigma}{\sigma_T} = \frac{\rho}{\rho_T} = \frac{\delta}{\delta_T} = \frac{p}{p_T} = e^{-k_h(H-h_T)} \]

- \( H_T = 11000 \) m = 11 km = 36089 ft
- \( k_h = 1.57688 \cdot 10^{-4} \) 1/m = 0.157688 1/km = 4.80634 \( \cdot 10^{-5} \) 1/ft
- \( \sigma_T = 0.297070 \)
- \( \rho_T = 0.3639 \) kg/m³
- \( \delta_T = 0.223356 \)
- \( p_T = 22632 \) Pa = 226.32 hPa = 0.22632 bar
Equations valid for both Troposphere and Stratosphere

speed of sound
\[ a = \sqrt{\gamma \cdot R} \cdot \sqrt{T} \]
\[ \sqrt{\gamma \cdot R} = 20.0468 \frac{1}{\sqrt{K}} \frac{m}{s} \]
\[ R = 287.053 \frac{J}{kg \cdot K} \]
\[ \gamma = 1.4 \]
\[ a_0 = 340.294 \text{ m/s} = 1225.06 \text{ km/h} = 661.48 \text{ kt} \]

dynamic viscosity
\[ \mu = \frac{\beta_s \cdot T^{3/2}}{T + S} \]
\[ \beta_s = 1.458 \cdot 10^{-6} \frac{kg}{m \cdot s \cdot \sqrt{K}} \]
\[ \mu_0 = 1.7894 \cdot 10^{-5} \frac{kg}{m \cdot s} \]
\[ S = 110.4 \text{ K} \]

kinematic viscosity
\[ v = \frac{\mu}{\rho} \]
\[ v_0 = 1.4607 \cdot 10^{-5} \frac{m^2}{s} \]

relative density
\[ \sigma = \frac{\rho}{\rho_0} \]

relative pressure
\[ \delta = \frac{p}{p_0} \]

relative temperature
\[ \theta = \frac{T}{T_0} \]

equation of state for a perfect gas
\[ \frac{p}{\rho} = R \cdot T \]
\[ R = 287.053 \frac{J}{kg \cdot K} \]
\[ \frac{\delta}{\sigma} = \theta \]

1.2 Height Scales and Conversions

geometric height
\( h \) as measured with a tape

géopotential height
\( H \) equivalent potential energy in a uniform gravitational field with \( g = 9.80665 \text{ m/s}^2 \)

pressure height
\( h_p \) equivalent height for given pressure in the ISA

density height
\( h_p \) equivalent height for given density in the ISA

<table>
<thead>
<tr>
<th>Altimeter Setting</th>
<th>Altimeter Reading</th>
</tr>
</thead>
<tbody>
<tr>
<td>1013 hPa</td>
<td>Standard pressure</td>
</tr>
<tr>
<td>QNH</td>
<td>Actual pressure at mean sea level</td>
</tr>
<tr>
<td>QFE</td>
<td>Actual pressure at the airfield</td>
</tr>
</tbody>
</table>

Definition: Flight Level = altitude above 1013 hPa pressure plain in ft / 100 ft

Example: FL 100 = 10000 ft / 100 ft
Conversion among different height definitions, pressure, density and temperature in the troposphere

**STEP 1:** We assume \( p_0 = 1013.25 \) hPa (standard condition) but \( T = T_{ISA} + \Delta T \).

\[
r_{\text{earth}} = 6.371 \cdot 10^6 \text{ m} = 6.371 \cdot 10^3 \text{ km} = 2.090 \cdot 10^7 \text{ ft}
\]

\[
g = g_0 \left( \frac{r_{\text{earth}}}{r_{\text{earth}} + h} \right)^2 \quad \text{with} \quad g_0 = 9.80665 \text{ m/s}^2
\]

---

**Fig. 1.3:** Conversion among different height definitions, pressure, density and temperature in the troposphere
STEP 2: If $p_0 \neq 1013.25$ hPa this can be accounted for as indicated in Fig. 1.4.

Fig. 1.4: Conversion among different height definitions, pressure, density and temperature in the troposphere

$$\Delta h \approx \Delta H = H = \frac{T_0 + \Delta T}{L} \left(1 - \delta^{\frac{1}{5.25588}}\right)$$

$$\Delta h \approx \frac{T_0 + \Delta T}{L} \left(1 - \left(\frac{QNH}{p_0}\right)^{\frac{1}{5.25588}}\right)$$

1.3 Aircraft Speed Definitions and Conversions

Indicated Airspeed $\rightarrow$ Calibrated Airspeed $\rightarrow$ Equivalent Airspeed $\rightarrow$ True Airspeed $\rightarrow$ Ground Speed

$V_I \rightarrow V_C \rightarrow V_E \rightarrow V \rightarrow V_G$
Conversion between different aircraft speed definitions

\[ V_c \rightarrow V_e: \quad V_e = a_0 \sqrt{58 \left[ \frac{1}{8} \left[ 1 + 0.2 \left( \frac{V_c}{a_0} \right)^2 \right]^{3.5} - 1 \right] + 1} - 1 \]

or

\[ V_e = V_c - \Delta V_c \quad \Delta V_c \text{ from Fig. 1.5} \]

\[ V_e \rightarrow V: \quad V = V_e / \sqrt{\sigma} \]

\[ V \rightarrow V_G: \quad V_G = \sqrt{V^2 + V_W^2 - 2 \cdot V \cdot V_W \cdot \cos \alpha} \]

\[ \text{Das schiefwinklige Dreieck} \]

\[ \text{Sinus-Satz} \]

\[ \sin a : \sin \beta : \sin \gamma = a : b : c \]

\[ a = \frac{b}{\sin \beta} \sin a = \frac{c}{\sin \gamma} \sin a \]

\[ b = \frac{a}{\sin \beta} \sin \beta = \frac{c}{\sin \gamma} \sin \beta \]

\[ c = \frac{a}{\sin \alpha} \sin \gamma = \frac{b}{\sin \beta} \sin \gamma \]

\[ \text{Cosinus-Satz} \]

\[ a^2 = b^2 + c^2 - 2 \cdot b \cdot c \cdot \cos \alpha \]

\[ b^2 = c^2 + a^2 - 2 \cdot a \cdot c \cdot \cos \beta \]

\[ c^2 = a^2 + b^2 - 2 \cdot a \cdot b \cdot \cos \gamma \]

\[ \text{GIECK 1981} \]
Fig. 1.5: Compressibility correction to calibrated airspeed [17]

\[
\begin{align*}
V^C & \rightarrow M = \left[ 1 - \left \{ 1 + \left[ 1 + \left( \frac{\rho}{\gamma A^2} \right) (0.0 + 1) \right] \left( \frac{\gamma}{1} \right) \right \} \right] \frac{S}{V^C} = W = \frac{\gamma}{1} \Delta V
\end{align*}
\]
1.4 Air Temperature

Measurements of the outside air temperature are done on board an aircraft with ventilated pitot air thermometers which are shielded against radiation. Since the aircraft is moving relative to the air mass, the temperature sensed by the air thermometer is higher than the ambient air temperature. If the air would be slowed down to stagnation conditions, the measured air temperature would be the stagnation or total air temperature $T_t (= \text{TAT})$. The total temperature caused by an adiabatic process is calculated from the static or outside air temperature $T (= \text{OAT})$ as a function of Mach number $M$:

$$\frac{T_t}{T} = 1 + \frac{1}{2} (\gamma - 1) \cdot M^2 = 1 + 0.2 \cdot M^2$$

In the ventilated pitot air thermometer the flow is collected but not totally slowed down to rest. The better the ventilation, the less will the thermometer respond against thermal radiation from the sun or local sources of heat. Ventilation is also necessary to ensure quick response to temperature changes of the air mass. However, ventilation prevents perfect stagnation conditions. The ventilated thermometer can only recover part of the stagnation temperature and shows the indicated air temperature $T_i (= \text{IAT})$. A recovery factor is defined as

$$k_r = \frac{T_i - T}{T_t - T}.$$  

For total recovery $k_r = 1$. Recovery factors range from about 0.8 to close to 1. A typical value might be $k_r = 0.97$. The indicated temperature is calculated from

$$\frac{T_i}{T} = 1 + \frac{1}{2} k_r (\gamma - 1) \cdot M^2 = 1 + 0.2 \cdot k_r \cdot M^2.$$  

More often it will be required to calculate the outside air temperature $T$ from the indicated temperature with

$$T = \frac{T_i}{1 + 0.2 \cdot k_r \cdot M^2}.$$
1.5 Rules of Thumb

<table>
<thead>
<tr>
<th>conversion</th>
<th>method</th>
<th>example</th>
</tr>
</thead>
<tbody>
<tr>
<td>ft → m</td>
<td>ft / 10 * 3</td>
<td>1000 ft = 1000 / 10 * 3 m = 300 m</td>
</tr>
<tr>
<td>NM → km</td>
<td>NM * 2 - 10%</td>
<td>1 NM ≈ 1 * 2 - 0.2 km = 1.8 km</td>
</tr>
</tbody>
</table>

1 hPa is equivalent to about 30 ft
3 Aircraft Drag and Drag Power

\[
\frac{E}{E_{\text{max}}} = \frac{2}{1 + C_L}\quad \text{with} \quad \overline{C_L} = \frac{C_L}{C_{L,\text{md}}}
\]

\(C_{L,\text{md}}\): lift coefficient in minimum drag flight i.e. flight at \(E_{\text{max}}\)

Fig. 3.1: L/D versus lift coefficient in normalized form
4 Powerplant Performance

4.1 Thrust and Efficiency

The product of the thrust of an engine and the forward velocity of the aircraft (relative to the earth, no wind) is equal to the thrust power - also called power available

\[ P_T = T V \]

Newton's second law yields the thrust equation (Fig. 4.1)

\[ T = \dot{m} \left( V_{jet} - V \right) \]

\( V_{jet} \) is the speed of the air leaving the device relative to the aircraft. The engine shaft power \( P_S \) is related to the thrust power \( P_T \) by the efficiency \( \eta_P \).

\[ P_T = \eta_P P_S \]

The propulsive efficiency \( \eta_P \) can be defined as

\[ \eta_P = \frac{\text{useful power available}}{\text{total power generated}} \]

The useful power available is power used to push the aircraft forward. The total power generated includes in addition to the power available also the power wasted in the air jet behind the aircraft (Fig. 4.2)

\[ \eta_P = \frac{T V}{T V + \frac{1}{2} \dot{m} \left( V_{jet} - V \right)^2} \]

Substituting the thrust equation yields

\[ \eta_P = \frac{\dot{m} \left( V_{jet} - V \right) V}{\dot{m} \left( V_{jet} - V \right) V + \frac{1}{2} \dot{m} \left( V_{jet} - V \right)^2} \]
Dividing numerator and denominator by $\dot{n}(V_{jet} - V)V$:

$$\eta_p = \frac{1}{1 + \frac{1}{2} \frac{V_{jet} - V}{V}} = \frac{1}{1 + \frac{1}{2} \frac{V_{jet}}{V} - \frac{1}{2}}.$$ 

The ideal efficiency of a propulsive device is finally

$$\eta_p = \frac{2}{1 + \frac{V_{jet}}{V}}.$$ 

Often, the efficiency is expressed by means of the velocity of the jet relative to the earth $w = V_{jet} - V$

$$\eta_p = \frac{1}{1 + \frac{w}{2V}}.$$
Fig. 4.1:
Reaction principle in propulsion
(Anderson 1999)

(a) Propulsive device

(b) Propulsive device produces thrust $T$ acting to the left.

(c) Air feels equal and opposite force $T$ acting to the right.

Fig. 4.2:
A propulsive device moving into stationary air with velocity $V_\infty$
(Anderson 1999)
4.2 Propeller Efficiency

The efficiency calculated above in the form \[ \eta_p = \frac{2}{1 + \frac{V_{jet}}{V}} \] was called *ideal efficiency* because it does not take into account:

- energy losses due to slipstream rotation,
- blade profile drag,
- non-uniform flow,
- compressibility effects
- propeller blockage due to a fuselage or a nacelle.

For performance calculations *measured propeller efficiencies* have to be used which account for all of the above losses. The efficiency is given as a function of the *advance ratio* (Fortschrittsgrad) of the propeller

\[ J = \frac{V}{n D}. \]

\( V \) is the aircraft speed, \( n \) is the propeller speed (number of propeller revolutions per second) and \( D \) is the propeller diameter. The advance ratio is proportional to the ratio of aircraft speed and propeller tip speed

\[ \left( \frac{V}{r \omega} \right)_{tip} = \frac{V}{(D/2)(2\pi n)} = \frac{V}{D \pi n} = \frac{J}{\pi}. \]

\( J \) is a similarity parameter for propeller performance, in the same category as the Mach number and the Reynolds number. However, propeller performance also depends on the propeller pitch angle \( \beta \) as shown in Fig. 4.3 and Fig. 4.4.
Fig. 4.3: Velocity and relative wind diagrams for a section of a revolving propeller: (a) at low aircraft speeds, (b) at high aircraft speeds (ANDERSON 1999)

Fig. 4.4: Variation in propeller efficiency $\eta_p$ with advance ratio $J$ for different pitch angles $\beta$ [36]
From Fig. 4.3 follows that the propeller has to be twisted. The propeller pitch angle has to be a function of the propeller radius. Propeller charts (like Fig. 4.4) must refer to a pitch angle \( \beta \) at a specific radius as e.g. 0.75 \( \cdot R \).

For light piston propeller aircraft with a fixed-pitch propeller, the propeller efficiency can be determined from a generic chart presented here as **Fig. 4.5**. The chart shows the efficiency \( \eta_p \) divided by the propeller efficiency at the design point \( (\eta_p)_\text{design} \) as a function of the advance ratio \( J \) divided by the advance ratio at the design point \( J_{\text{design}} \)

\[
\frac{J}{J_{\text{design}}} = \frac{V}{V_{\text{design}}} \cdot \frac{\eta_{\text{design}}}{\eta} .
\]

**Fig. 4.5:** General chart for fixed-pitch propeller efficiency [28]

An example is presented on how the general chart (Fig. 4.5) is used:

A propeller is designed for cruise speeds of 100 kt and 2400 rpm (revolutions per minute) achieving an efficiency \( (\eta_p)_\text{design} = 0.8 \). What is the efficiency \( \eta_p \) during climb with an aircraft speed of 80 kt and 2500 rpm?

Answer:
\[ \frac{J}{J_{\text{design}}} = \frac{V}{V_{\text{design}}} \cdot \frac{n_{\text{design}}}{n} = \frac{80}{100} \cdot \frac{2400}{2500} = 0.768 \]

and with Fig. 4.5

\[ \eta_p = \left( \frac{\eta_p}{\text{design}} \right) \cdot \frac{\eta_p}{\eta_p_{\text{design}}} = 0.8 \cdot 0.87 = 0.7 \]

Thrust is calculated from

\[ T = \frac{\eta_p \cdot P_s}{V} \]

The static thrust at the beginning of the take-off run can not be calculated from the above equation because at \( V = 0 \) also \( J = 0 \) and hence \( \eta_p = 0 \). So we have a case 0/0. "As a rough approximation it can be assumed that the static thrust is about equal to the thrust at 50 knots for the fixed-pitch propeller." [28]
5 Level, Climbing and Descending Flight

5.1 Climb and Climb Schedules

For positive climb gradients the speed for best rate of climb is above that speed for best climb angle (Fig. 5.1). With decreasing climb performance, the speeds for best angle and best rate of climb come together. Finally, in the range of negative rate of climb, the speed for best angle is above that for best descent rate. Hence, in descending flight, range is maximized with a speed above that for maximum time aloft.

Fig. 5.1: Maximum rate of climb and maximum climb angle for different climb performance (LUFTansa 1988)

Fig. 5.2: A typical climb schedule and resulting rate of climb (LUFTansa 1988)
A typical climb schedule (Fig. 5.2-left) consists of three parts:
1. In the lower part of the troposphere, the climb is performed at constant IAS. This is done up to a point where a certain Mach number is reached.
2. It follows a constant Mach number climb in the troposphere results in a decreasing true air speed.
3. In the stratosphere, the continued constant Mach number climb results in a flight at constant true air speed.

In part 1, the climb rate is about 25% below the theoretical values for unaccelerated climb. In part 2, the climb rate is about 9% above the theoretical values for unaccelerated climb. During transition from constant IAS flight to constant Mach number flight, an increase in climb rate of up to 30% can be observed LUFTHANSA 1988. In part 3, climb with constant true air speed results in an unaccelerated climb.

Fig. 5.3:
Influence of altitude on rate of climb
(LUFTHANSA 1988)

Fig. 5.3 shows the rate of climb as a function of true air speed for different altitudes. With increasing altitude, the thrust decreases at a higher rate than the drag. On account of this, the excess thrust and the rate of climb drop. Fig. 5.3 indicates also the speed for a maximum rate of climb. This optimum climb schedule is approximated with a constant IAS / constant Mach number climb schedule.

Fig. 5.4 shows an increase in the speed for best rate of climb with increasing weight.
5.2 Time to Climb

Assume a linear variation of the rate of climb \( ROC = V_v \) within small altitude intervals \( \Delta h \). The time to climb through this altitude interval \( \Delta h \) is \( \Delta t \). The time to climb to an altitude \( h = j \cdot \Delta h \)

is \( t = \sum_{j=1}^{i} \Delta t_j \) with \( \Delta t = \frac{1}{A_j} \cdot \ln \left( \frac{V_{v_{i+1}}}{V_{v_i}} \right) \) and \( A_j = \frac{V_{v_{i+1}} - V_{v_i}}{\Delta h} = \frac{V_{v_{i+1}} - V_{v_i}}{h_{i+1} - h_i} \).

A detailed derivation of the Equation for \( \Delta t \) from Chapter 5 Page 23 is given in Appendix A.

5.3 Approximate Climb Calculation

The approximate climb calculation assumes a linear variation of the rate of climb \( ROC = V_v \) within the whole climb segment. This assumption is illustrated in Fig. 5.5.

\[
ROC = ROC_0 - \frac{ROC_0}{h_{abs}} \cdot h = ROC_0 \left( 1 - \frac{h}{h_{abs}} \right) \quad \text{or} \quad \frac{ROC}{ROC_0} = 1 - \frac{h}{h_{abs}}
\]
\[ t_{\text{CLB}} = \frac{1}{A} \ln \left( \frac{\text{ROC}}{\text{ROC}_0} \right) \quad \text{with} \quad A = \frac{\text{ROC} - \text{ROC}_0}{h} = \frac{-\text{ROC}_0}{h_{\text{abs}}} \]

\[ t_{\text{CLB}} = -\frac{h_{\text{abs}}}{\text{ROC}_0} \ln \left( \frac{\text{ROC}}{\text{ROC}_0} \right) \quad \text{or} \quad t_{\text{CLB}} = -\frac{h_{\text{abs}}}{\text{ROC}_0} \ln \left( 1 - \frac{h}{h_{\text{abs}}} \right) . \]

**Fig. 5.5:**
Assumptions for the approximate climb calculation

![Diagram](image-url)
6 Stall, Speed Stability, Turning Performance

It might be useful to introduce the standard instruments of light aircraft at this point (Fig. 6.1).

Fig. 6.1: Standard instruments of light aircraft (KÜHR 1987)

Table 6.1: Standard instruments and indicated flight parameters

<table>
<thead>
<tr>
<th>Instrument</th>
<th>Parameter</th>
<th>Formula</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fahrtmesser</td>
<td>airspeed indicator</td>
<td>$V_{IAS}$</td>
<td>IAS</td>
</tr>
<tr>
<td>künstlicher Horizont</td>
<td>artificial horizon</td>
<td>$\Phi, \Theta$</td>
<td>bank angle, pitch attitude</td>
</tr>
<tr>
<td>Höhenmesser</td>
<td>altimeter</td>
<td>$h$</td>
<td>altitude</td>
</tr>
<tr>
<td>Kurven-Koordinator (Wendezeiger)</td>
<td>turn and bank (or slip)</td>
<td>$\Omega$</td>
<td>rate of turn</td>
</tr>
<tr>
<td></td>
<td>indicator</td>
<td>at markings: 3°/s</td>
<td></td>
</tr>
<tr>
<td>Kurskreisel</td>
<td>directional giro</td>
<td>$dh/dt = V_v = ROC$</td>
<td>heading</td>
</tr>
<tr>
<td>Variometer</td>
<td>vertical speed indicator</td>
<td></td>
<td>vertical speed</td>
</tr>
</tbody>
</table>
## References

<table>
<thead>
<tr>
<th>Reference</th>
<th>Description</th>
</tr>
</thead>
</table>
Appendix

A Derivations

\[ H = \frac{r_{earth} \cdot h}{r_{earth} + h} \]

geopotential height calculated from geometric height

By definition:

\[ \text{potential energy} = m \cdot g \cdot H = \int_0^h m \cdot g(y) \cdot dy \]

(1)

Newton:

\[ F = G \cdot \frac{m \cdot m_{earth}}{r^2} = m \cdot g(h) \]

\[ g(h) = g_0 \left( \frac{r_{earth}}{r_{earth} + h} \right)^2 \]

(2)

from (1) and (2):

\[ H = \frac{1}{g_0} \int_0^h g_0 \left( \frac{r_{earth}}{r_{earth} + y} \right)^2 dy \]

\[ H = r_{earth} \int_0^h \frac{1}{(r_{earth} + y)^2} dy \]

\[ H = -r_{earth} \left[ \frac{1}{r_{earth} + y} \right]_0^h = -r_{earth} \left[ \frac{1}{r_{earth} + h} - \frac{1}{r_{earth}} \right] = -r_{earth} \left[ \frac{r_{earth} - (r_{earth} + h)}{(r_{earth} + h) \cdot r_{earth}} \right] \]

\[ H = \frac{r_{earth} \cdot h}{r_{earth} + h} \]

\[ H = h_p \cdot \frac{T_0 + \Delta T}{L} \]

gopotential height calculated from pressure height in the troposphere

\[ p = p_0 \cdot \left( 1 - \frac{L}{T_0 + \Delta T} \cdot H \right)^{5.2588} \]

for an atmosphere with \( L, R, g_0 \) from ISA but with

\[ T(h = 0) = T_0 + \Delta T \]
\[ p = p_0 \cdot \left(1 - \frac{L}{T_0} \cdot h_p\right)^{5.25588} \quad \text{by definition} \quad (2) \]

\[ p = p_0 \cdot \left(1 - \frac{L}{T_0 + \Delta T} \cdot H\right)^{5.25588} = p_0 \cdot \left(1 - \frac{L}{T_0} \cdot h_p\right)^{5.25588} \quad \text{from (1) and (2)} \]

\[ \left(1 - \frac{L}{T_0 + \Delta T} \cdot H\right)^{5.25588} = \left(1 - \frac{L}{T_0} \cdot h_p\right)^{5.25588} \]

\[ \frac{L}{T_0 + \Delta T} \cdot H = \frac{L}{T_0} \cdot h_p \]

\[ H = h_p \cdot \frac{T_0 + \Delta T}{T_0} \]

\[
\begin{align*}
\frac{E}{E_{\text{max}}} & = \frac{2}{1 + \overline{C_L}} \\
\text{with } \overline{C_L} & = \frac{C_L}{C_{L,\text{md}}} \\
\end{align*}
\]

\[
\begin{align*}
\frac{C_D}{C_{D,\text{md}}} & = \frac{C_{D0} + \frac{C_L^2}{\pi \cdot A \cdot e}}{C_{D0} + \frac{C_{L,\text{md}}^2}{\pi \cdot A \cdot e}} \\
C_{D,\text{md}} : \text{ minimum drag flight i.e. flight at } E_{\text{max}}}
\end{align*}
\]

\[
\begin{align*}
\frac{C_D}{C_{D,\text{md}}} & = \frac{C_{D0} + \frac{C_L^2}{\pi \cdot A \cdot e}}{2 \cdot C_{D0}} = \frac{1}{2} + \frac{C_L^2}{2 \cdot C_{D0} \cdot \pi \cdot A \cdot e} = \frac{1}{2} \left(1 + \frac{C_L^2}{C_{D0} \cdot \pi \cdot A \cdot e}\right) = \frac{1}{2} \left(1 + \frac{\frac{C_L^2}{C_{L,\text{md}}^2}}{\frac{C_{D,\text{md}}^2}{\pi \cdot A \cdot e} \cdot \pi \cdot A \cdot e}\right) \\
& = \frac{1}{2} \left(1 + \left(\frac{C_L}{C_{L,\text{md}}}\right)^2\right) = \frac{1}{2} \left(1 + \overline{C_L}^2\right) \\
\end{align*}
\]

\[
\begin{align*}
\frac{E}{E_{\text{max}}} & = \frac{\frac{C_L}{C_{L,\text{md}}}}{C_{D,\text{md}}} = \frac{C_{D,\text{md}}}{C_D} \cdot \frac{C_L}{C_{L,\text{md}}} = \frac{2}{1 + \overline{C_L}^2} \cdot \overline{C_L} = \frac{2}{\overline{C_L} + \overline{C_L}} \\
\end{align*}
\]
A more detailed derivation of Equation [5.3-25] from chapter 5 page 23.

Integrating time to climb from ROC:

\[
\Delta t = \frac{1}{A_i} \cdot \ln \left( \frac{V_{v,i+1}}{V_{v,i}} \right)
\]

Assuming a linear variation of ROC within interval \(i\):

\[
V_v = \frac{V_{v,i+1} - V_{v,i}}{\Delta h} \cdot (h - h_i) + V_{v,i}
\]

\[
A_i = \frac{V_{v,i+1} - V_{v,i}}{\Delta h} = \frac{V_{v,i+1} - V_{v,i}}{h_{i+1} - h_i}
\]

Introducing a new variable:

\[h' = h - h_i\]

\[
t = \int_{h_i}^{h_{i+1}} \frac{1}{A_i \cdot (h - h_i) + V_{v,i}} dh
\]

Substitution:

\[
t = \int_{h_i}^{h_{i+1}} \frac{1}{A_i \cdot h' + V_{v,i}} \cdot dh' dh
\]

Solving the integral:

\[
t = \frac{1}{A_i} \left[ \ln \left( A_i \cdot h' + V_{v,i} \right) \right]_{h_i}^{h_{i+1}} = \frac{1}{A_i} \left[ \ln \left( A_i \cdot h' + V_{v,i} \right) \right]_{h_i}^{h_{i+1}}
\]

This time \(t\) - also called \(\Delta t\) - is the time required to climb from an altitude \(h_i\) to an altitude \(h_{i+1}\).

\[
\Delta t = \frac{1}{A_i} \cdot \ln \left( \frac{V_{v,i+1}}{V_{v,i}} \right)
\]