7.5 SUMMARY OF IMPORTANT RESULTS

Table 7.5-1 Summary of Range and Endurance Equations

		Turbofan / turbojet		Piston prop
Range for schedules:	1	$R = \frac{2EV_1}{cg}\left[1 - \sqrt{\frac{m_2}{m_1}}\right]$	1	$R = \frac{\eta_P E}{c'g}\log_e\left(\frac{m_1}{m_2}\right)$
	2	$R = \frac{VE}{cg}\log_e\left(\frac{m_1}{m_2}\right)$	2	
	3	$R = \frac{2E_{max}V}{gc}\arctan\left(\frac{\sqrt{B_3}(m_1 - m_2)}{B_3 + m_1 m_2}\right)$	3	$R = \frac{2E_{max}\eta_p}{gc'}\arctan\left(\frac{\sqrt{B_3}(m_1 - m_2)}{B_3 + m_1 m_2}\right)$
Endurance for schedules:	1	$t = \frac{E}{cg}\log_e\left(\frac{m_1}{m_2}\right)$	1	$t = \frac{2\eta_p E}{c'gV_1}\left[\sqrt{\frac{m_1}{m_2}} - 1\right]$
	2		2	$t = \frac{\eta_P E}{c'gV}\log_e\left(\frac{m_1}{m_2}\right)$
	3	$t = \frac{2E_{max}}{gc}\arctan\left(\frac{\sqrt{B_3}(m_1 - m_2)}{B_3 + m_1 m_2}\right)$	3	$t = \frac{2E_{max}\eta_p}{gc'V}\arctan\left(\frac{\sqrt{B_3}(m_1 - m_2)}{B_3 + m_1 m_2}\right)$

Flight schedules:

1. Flight at constant *altitude* and constant *lift coefficient*
2. Flight at constant *airspeed* and constant *lift coefficient*
3. Flight at constant *altitude* and constant *airspeed*

Notation:

m_1 is initial (start of cruise / endurance) mass and m_2 is the final mass

V_1 is the TAS at the start of the cruise / endurance

$$B_3 = \frac{C_{D_o}\rho^2 S^2 V^4 \pi A e}{4g^2}$$

B.2 DETERMINATION OF STILL AIR RANGE EQUATIONS FOR ALTERNATIVE FLIGHTS PROGRAMS

The range equation may be evaluated for alternative flight programs. The following programs are considered for turbo-jet and piston aircraft.

1. Cruise at constant *altitude* and constant *lift coefficient*
2. Cruise at constant *airspeed* and constant *lift coefficient*
3. Cruise at constant *altitude* and constant *airspeed*

Note that flight at constant C_L implies constant lift-to-drag ratio, E. In all of the derivations the SFC is assumed constant. The parabolic drag polar is used in determining the range equation for flight program three.

B.2.1 Determination of the Range Equation for a Turbo-jet flying at Constant Altitude and Constant Lift Coefficient (first flight program)

From equation [7.1-7] of chapter 7, the still air range, R for a turbo- jet is given by:

$$R = -\int_{m_1}^{m_2} \frac{VE}{cmg} dm$$

$$R = \frac{-E}{cg}\int_{m_1}^{m_2} \frac{V}{m}\, dm \qquad \text{(Note } E \text{ is constant as } C_L \text{ is constant.)}$$

Now for level flight: $C_L = \dfrac{mg}{\frac{1}{2}\rho V^2 S}$ Therefore: $V = \sqrt{\dfrac{2mg}{\rho S C_L}}$

$$R = \frac{-E}{cg}\int_{m_1}^{m_2} \sqrt{\frac{2\,mg}{\rho S C_L}}\frac{1}{m}\, dm$$

$$= \frac{-E}{cg}\sqrt{\frac{2g}{\rho S C_L}}\int_{m_1}^{m_2} m^{-1/2}\, dm \qquad \text{(Note } \rho \text{ is constant as } h \text{ is constant.)}$$

$$= \frac{-E}{cg}\sqrt{\frac{2g\,m_1}{\rho S C_L}}\,\frac{1}{\sqrt{m_1}}\left(\frac{1}{1/2}\right)\left[\sqrt{m_2} - \sqrt{m_1}\right]$$

$$= \frac{-2E}{cg}\sqrt{\frac{2gm_1}{\rho S C_L}}\left[\frac{\sqrt{m_2}}{\sqrt{m_1}} - 1\right]$$

Finally: $$R = \frac{2EV_1}{cg}\left[1 - \sqrt{\frac{m_2}{m_1}}\right] \qquad \text{----} \qquad \text{[B-2]}$$

where V_1 is the *start-of-cruise* TAS.

B.2.2 Determination of the Still Air Range Equation for a Turbo-jet flying at Constant TAS and Constant Lift Coefficient (second flight program)

The second flight program is usually called the *cruise-climb,* as altitude is not constant.

$$R = -\int_{m_1}^{m_2} \frac{VE}{cmg}\,dm$$

$$= -\frac{VE}{cg}\int_{m_1}^{m_2} \frac{1}{m}\,dm \qquad \text{(Both } V \text{ and } E \text{ are constant.)}$$

$$= \frac{VE}{cg}\left[\log_e m_1 - \log_e m_1\right]$$

$$R = \frac{VE}{cg}\log_e\left(\frac{m_1}{m_2}\right) \qquad \text{----} \qquad \text{[B-3]}$$

This equation is known as the *Breguet Range Equation.*

B.2.3 Determination of the Still Air Range Equation for a Turbo-jet flying at Constant Altitude and Constant TAS (third flight program)

The still air range for a turbo- jet is given by:

$$R = -\int_{m_1}^{m_2} \frac{VE}{cmg}\,dm$$

For this application, it is easier to start with the equivalent expression:

$$R = -\int_{m_1}^{m_2} \frac{V}{cD}\, dm$$

Now: $$\frac{V}{cD} = \frac{V}{c\left(\dfrac{C_{D_o}\rho S V^2}{2} + \dfrac{2m^2 g^2}{\pi A e \rho S V^2}\right)}$$

$$\frac{V}{cD} = \frac{\left(\dfrac{\pi A e \rho S V^3}{2g^2 c}\right)}{\left(\dfrac{C_{D_o}\rho^2 S^2 V^4 \pi A e}{4g^2}\right) + m^2}$$

Put: $$A_3 = \frac{\pi A e \rho S V^3}{2g^2 c}$$ and: $$B_3 = \frac{C_{D_o}\rho^2 S^2 V^4 \pi A e}{4g^2}$$

Hence: $$R = -\int_{m_1}^{m_2} \frac{A_3}{B_3 + m^2}\, dm$$

$$= \frac{A_3}{\sqrt{B_3}}\left[\arctan\frac{m_1}{\sqrt{B_3}} - \arctan\frac{m_2}{\sqrt{B_3}}\right]$$

Put $$\tan\alpha = \frac{m_1}{\sqrt{B_3}}$$ where: $$\alpha = \arctan\frac{m_1}{\sqrt{B_3}}$$

and: $$\tan\beta = \frac{m_2}{\sqrt{B_3}}$$ where: $$\beta = \arctan\frac{m_2}{\sqrt{B_3}}$$

Now: $$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

$$(\alpha-\beta) = \arctan\left(\frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}\right)$$

$$(\alpha-\beta) = \arctan\left(\frac{\dfrac{m_1}{\sqrt{B_3}} - \dfrac{m_2}{\sqrt{B_3}}}{1 + \dfrac{m_1 m_2}{B_3}}\right)$$

$$= \arctan\left(\frac{\sqrt{B_3}(m_1 - m_2)}{B_3 + m_1 m_2}\right)$$

Also: $$\frac{A_3}{\sqrt{B_3}} = \frac{\pi A e \rho S V^3}{2g^2 c}\sqrt{\frac{4g^2}{C_{D_o}\rho^2 S^2 V^4 \pi A e}} = \sqrt{\frac{\pi A e}{C_{D_o}}}\frac{V}{gc} = 2E_{max}\frac{V}{gc}$$

Finally: $$R = \frac{2E_{max}V}{gc}\arctan\left(\frac{\sqrt{B_3}(m_1 - m_2)}{B_3 + m_1 m_2}\right) \quad \text{---} \quad [B-4]$$

where: $$B_3 = \frac{C_{D_o}\rho^2 S^2 V^4 \pi A e}{4g^2}$$

B.2.4 Determination of the Still Air Range Equation for a Piston-prop flying at Constant Altitude and Constant Lift Coefficient or Constant TAS and Constant Lift Coefficient (first and second flight programs)

From equation [7.2-1] of chapter 7, the still air range for a piston-prop is given by:

$$R = -\int_{m_1}^{m_2} \frac{\eta_p E}{c' mg}\, dm$$

$$= -\frac{\eta_p E}{c' g}\int_{m_1}^{m_2} \frac{1}{m}\, dm \qquad (E \text{ is constant})$$

$$R = \frac{\eta_P E}{c' g}\log_e\left(\frac{m_1}{m_2}\right) \quad \text{---} \quad [B-5]$$

This equation is the Breguet range equation for piston-propeller aircraft.

B.2.5 Determination of the Still Air Range Equation for a Piston-prop flying at Constant Altitude and Constant TAS (third flight program)

From equation [7.2-1] of chapter 7, the still air range for a piston-prop is given by:

$$R = -\int_{m_1}^{m_2} \frac{\eta_p E}{c' mg} dm$$

For this application, it is easier to start with an equivalent expression.

$$R = -\int_{m_1}^{m_2} \frac{\eta_p}{c' D} dm$$

Now: $$\frac{\eta_p}{c' D} = \frac{\eta_p}{c'\left(\dfrac{C_{D_o}\rho S V^2}{2} + \dfrac{2m^2 g^2}{\pi A e \rho S V^2}\right)}$$

$$= \frac{\left(\dfrac{\pi A e \rho S V^2 \eta_p}{2g^2 c'}\right)}{\left(\dfrac{C_{D_o}\rho^2 S^2 V^4 \pi A e}{4g^2}\right) + m^2}$$

Put: $A_4 = \dfrac{\pi A e \rho S V^2 \eta_p}{2g^2 c}$ and: $B_3 = \dfrac{C_{D_o}\rho^2 S^2 V^4 \pi A e}{4g^2}$

Hence: $$R = -\int_{m_1}^{m_2} \frac{A_4}{B_3 + m^2} dm$$

$$= \frac{A_4}{\sqrt{B_3}}\left[\arctan\frac{m_1}{\sqrt{B_3}} - \arctan\frac{m_2}{\sqrt{B_3}}\right]$$

Now: $$\frac{A_4}{\sqrt{B_3}} = \frac{\pi A e \rho S V^2 \eta_p}{2g^2 c'}\sqrt{\frac{4g^2}{C_{D_o}\rho^2 S^2 V^4 \pi A e}} = \sqrt{\frac{\pi A e}{C_{D_o}}}\frac{\eta_p}{gc'} = 2E_{max}\frac{\eta_p}{gc'}$$

In appendix B.2.3 it was shown that:

$$\arctan\frac{m_1}{\sqrt{B_3}} - \arctan\frac{m_2}{\sqrt{B_3}} = \arctan\left(\frac{\sqrt{B_3}(m_1 - m_2)}{B_3 + m_1 m_2}\right).$$

Finally: $$R = \frac{2E_{max}\eta_p}{gc'}\arctan\left(\frac{\sqrt{B_3}(m_1 - m_2)}{B_3 + m_1 m_2}\right) \quad \text{---} \quad [B-6]$$

where: $$B_3 = \frac{C_{D_o}\rho^2 S^2 V^4 \pi A e}{4g^2}$$

B.3 DETERMINATION OF ENDURANCE EQUATIONS FOR ALTERNATIVE FLIGHTS PROGRAMS

The flight time (endurance) may be evaluated for alternative flight programs. The following programs are considered for turbo-jet and piston aircraft.

1. Flight at constant *altitude* and constant *lift coefficient*
2. Flight at constant *airspeed* and constant *lift coefficient*
3. Flight at constant *altitude* and constant *airspeed*

Note that flight at constant C_L implies constant lift-to-drag ratio, E. In all of the derivations the SFC is assumed constant. The parabolic drag polar is used in determining the endurance equation for flight program three.

B.3.1 Determination of the Endurance for a Turbo-jet flying at Constant Altitude and Constant Lift Coefficient or Constant TAS and Constant Lift Coefficient (first and second flight programs)

From equation [7.3-2] of chapter 7, the endurance (flight time) for a turbo- jet is given by:

$$t = -\int_{m_1}^{m_2}\frac{E}{cmg}dm$$

$$t = -\frac{E}{cg}\int_{m_1}^{m_2} \frac{1}{m}\,dm \qquad \text{(Constant } C_L \text{ implies constant } E\text{)}$$

Hence: $$t = \frac{E}{cg}\log_e\left(\frac{m_1}{m_2}\right) \qquad \text{---- [B-7]}$$

B.3.2 Determination of the Endurance Equation for a Turbo-jet flying at Constant Altitude and Constant TAS

The endurance (loiter time) for a turbo- jet is given by:

$$t = -\int_{m_1}^{m_2} \frac{E}{cmg}\,dm$$

For this application, it is easier to start with the equivalent expression:

$$t = -\int_{m_1}^{m_2} \frac{1}{cD}\,dm$$

Now: $$\frac{1}{cD} = \frac{1}{c\left(\dfrac{C_{D_o}\rho S V^2}{2} + \dfrac{2m^2g^2}{\pi A e \rho S V^2}\right)}$$

$$= \frac{\left(\dfrac{\pi A e \rho S V^2}{2g^2c}\right)}{\left(\dfrac{C_{D_o}\rho^2 S^2 V^4 \pi A e}{4g^2}\right) + m^2}$$

Put: $A_5 = \dfrac{\pi A e \rho S V^2}{2g^2c}$ and: $B_3 = \dfrac{C_{D_o}\rho^2 S^2 V^4 \pi A e}{4g^2}$

Hence: $$t = -\int_{m_1}^{m_2} \frac{A_5}{B_3 + m^2}\,dm$$

$$t = \frac{A_5}{\sqrt{B_3}}\left[\arctan\frac{m_1}{\sqrt{B_3}} - \arctan\frac{m_2}{\sqrt{B_3}}\right]$$

Now: $$\frac{A_5}{\sqrt{B_3}} = \frac{\pi A e \rho S V^2}{2g^2 c}\sqrt{\frac{4g^2}{C_{D_o}\rho^2 S^2 V^4 \pi A e}} = \sqrt{\frac{\pi A e}{C_{D_o}}}\frac{1}{gc} = \frac{2E_{max}}{gc}$$

In appendix B.2.3 it was shown that:

$$\arctan\frac{m_1}{\sqrt{B_3}} - \arctan\frac{m_2}{\sqrt{B_3}} = \arctan\left(\frac{\sqrt{B_3}(m_1 - m_2)}{B_3 + m_1 m_2}\right)$$

Finally: $$t = \frac{2E_{max}}{gc}\arctan\left(\frac{\sqrt{B_3}(m_1 - m_2)}{B_3 + m_1 m_2}\right) \quad \text{--- [B-8]}$$

where: $$B_3 = \frac{C_{D_o}\rho^2 S^2 V^4 \pi A e}{4g^2}$$

B.3.3 Determination of the Endurance Equation for a Piston-prop flying at Constant Altitude and Constant Lift Coefficient (first flight program)

The endurance time, for a piston-prop, may be determined using equation [7.4-2] of chapter 7:

$$t = -\int_{m_1}^{m_2}\frac{\eta_P E}{c' g m V}\,dm$$

$$t = \frac{-\eta_P E}{c' g}\int_{m_1}^{m_2}\frac{1}{mV}\,dm$$

(Note E is constant as C_L is constant.)

Now for level flight: $$C_L = \frac{mg}{\frac{1}{2}\rho V^2 S}$$ Therefore: $$V = \sqrt{\frac{2mg}{\rho S C_L}}$$

$$t = \frac{-\eta_P E}{c' g}\int_{m_1}^{m_2}\sqrt{\frac{\rho S C_L}{2mg}}\frac{1}{m}\,dm$$

$$t = \frac{-\eta_P E}{c' g}\sqrt{\frac{\rho S C_L}{2g}}\int_{m_1}^{m_2} m^{-3/2}\, dm \qquad \text{(Note } \rho \text{ is constant as } h \text{ is constant.)}$$

$$= \frac{-\eta_P E}{c' g}\sqrt{\frac{\rho S C_L}{2g m_1}}\sqrt{m_1}\left(\frac{1}{-\frac{1}{2}}\right)\left[\frac{1}{\sqrt{m_2}} - \frac{1}{\sqrt{m_1}}\right]$$

$$= \frac{2\eta_P E}{c' g}\sqrt{\frac{\rho S C_L}{2g m_1}}\left[\frac{\sqrt{m_1}}{\sqrt{m_2}} - 1\right]$$

Finally:

$$t = \frac{2\eta_P E}{c' g V_1}\left[\sqrt{\frac{m_1}{m_2}} - 1\right] \qquad \text{--- [B-9]}$$

where V_1 is the TAS at the start of the endurance.

B.3.4 Determination of the Endurance Equation for a Piston-prop flying at Constant TAS and Constant Lift Coefficient (second flight program)

The endurance time, for a piston-prop is given by:

$$t = -\int_{m_1}^{m_2} \frac{\eta_P E}{c' g m V}\, dm$$

$$t = -\frac{\eta_P E}{c' g V}\int_{m_1}^{m_2} \frac{1}{m}\, dm \qquad \text{(Constant airspeed and constant } C_L\text{)}$$

$$t = \frac{\eta_P E}{c' g V}\log_e\left(\frac{m_1}{m_2}\right) \qquad \text{--- [B-10]}$$

As height is not constant this means that the aircraft flies a cruise climb.

B.3.5 Determination of the Endurance Equation for a Piston-prop flying at Constant Altitude and Constant TAS (third flight program)

From equation [7.4-2] of chapter 7, the endurance (loiter time) for a piston-prop is given by:

$$t = -\int_{m_1}^{m_2} \frac{\eta_p E}{c' mgV} dm$$

For this application, it is easier to start with the equivalent expression:

$$t = -\int_{m_1}^{m_2} \frac{\eta_p}{c' DV} dm$$

Now:

$$\frac{\eta_p}{c' DV} = \frac{\eta_p}{c' V\left(\dfrac{C_{D_o}\rho S V^2}{2} + \dfrac{2m^2 g^2}{\pi A e \rho S V^2}\right)}$$

$$= \frac{\left(\dfrac{\pi A e \rho S V \eta_p}{2g^2 c'}\right)}{\left(\dfrac{C_{D_o}\rho^2 S^2 V^4 \pi A e}{4g^2}\right) + m^2}$$

Put: $A_6 = \dfrac{\pi A e \rho S V \eta_p}{2g^2 c}$ and: $B_3 = \dfrac{C_{D_o}\rho^2 S^2 V^4 \pi A e}{4g^2}$

Hence:

$$t = -\int_{m_1}^{m_2} \frac{A_6}{B_3 + m^2} dm$$

$$= \frac{A_6}{\sqrt{B_3}}\left[\arctan\frac{m_1}{\sqrt{B_3}} - \arctan\frac{m_2}{\sqrt{B_3}}\right]$$

Now:

$$\frac{A_6}{\sqrt{B_3}} = \frac{\pi A e \rho S V \eta_p}{2g^2 c'}\sqrt{\frac{4g^2}{C_{D_o}\rho^2 S^2 V^4 \pi A e}} = \sqrt{\frac{\pi A e}{C_{D_o}}}\,\frac{\eta_p}{gc'V} = 2E_{max}\frac{\eta_p}{gc'V}$$

In appendix B.2.3 it was shown that:

$$\arctan\frac{m_1}{\sqrt{B_3}} - \arctan\frac{m_2}{\sqrt{B_3}} = \arctan\left(\frac{\sqrt{B_3}(m_1 - m_2)}{B_3 + m_1 m_2}\right).$$

Finally:

$$t = \frac{2E_{max}\eta_p}{gc'V}\arctan\left(\frac{\sqrt{B_3}(m_1 - m_2)}{B_3 + m_1 m_2}\right) \quad \text{---} \quad \text{[B-11]}$$

where:

$$B_3 = \frac{C_{D_o}\rho^2 S^2 V^4 \pi A e}{4g^2}$$

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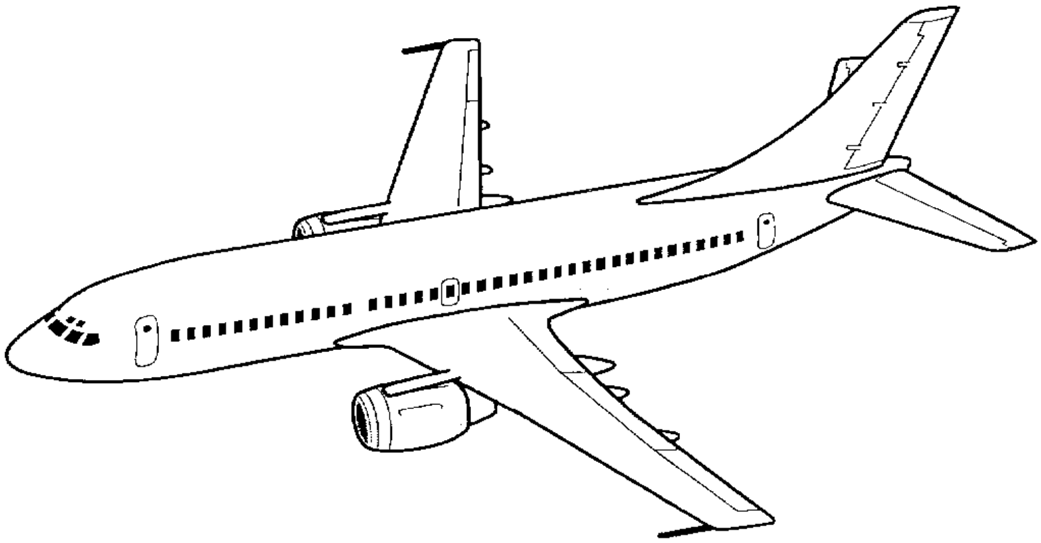

FLIGHT MECHANICS

COURSE ME 4726

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