Optimality of constant-Mach constant-altitude cruise for high-speed commercial aircraft

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EWADE 2011
Napoli, 24-27 May 2011

Objectives

— **Optimality of standard const-M, const-h procedure**

— **Optimal cruise** $(M^*(W),\, h^*(W),\, R_{max})$

— **Best standard cruise** $(M_{best},\, h_{best},\, R_{best})$

— **Comparison** $(\dfrac{R_{best}}{R_{max}})$

Formulation

Equations of motion

$$T = D$$
$$L = W$$
$$\frac{\mathrm{d}r}{\mathrm{d}t} = V$$
$$\frac{1}{g}\frac{\mathrm{d}W}{\mathrm{d}t} = -cT$$

Aerodynamic model

$$C_D = C_D(M, C_L)$$

Propulsive model

$$c = \frac{a(h)}{L_H} C_C(M)$$

Formulation (2)

Range

$$R = \int_{t_i}^{t_f} V \, \mathrm{d}t = -\frac{1}{g} \int_{W_i}^{W_f} \frac{V}{c\,T} \, \mathrm{d}W = \frac{1}{g} \int_{W_f}^{W_i} \frac{a\ M}{c\ D} \, \mathrm{d}W$$

$$D = q_0 \delta M^2 C_D, \qquad C_L = \frac{W}{q_0 \delta M^2}, \qquad \delta = \frac{p}{p_0}, \qquad q_0 = \frac{1}{2} \kappa p_0 S$$

$$S_R = \frac{a(h)\ M}{g\ c(M,\delta)\ D(M,\delta,W)}$$

$$R = \int_{W_f}^{W_i} S_R\,(M,\delta,W)\ \mathrm{d}W$$

Optimal cruise: find the optimal control laws $M^*(W)$ and $\delta^*(W)$ that maximize range

Euler-Lagrange equations

$$\begin{aligned}
\frac{\partial S_R}{\partial M} - \frac{\mathrm{d}}{\mathrm{d}W}\left(\frac{\partial S_R}{\partial \dot{M}}\right) &= 0 \\
\frac{\partial S_R}{\partial \delta} - \frac{\mathrm{d}}{\mathrm{d}W}\left(\frac{\partial S_R}{\partial \dot{\delta}}\right) &= 0
\end{aligned}$$

$$\begin{aligned}
\frac{\partial S_R}{\partial M} &= 0 \\
\frac{\partial S_R}{\partial \delta} &= 0
\end{aligned}$$

$$\begin{aligned}
D\left(c - M\frac{\partial c}{\partial M}\right) - cM\frac{\partial D}{\partial M} &= 0 \\
\frac{\partial D}{\partial \delta} &= 0
\end{aligned}$$

Dimensionless Euler-Lagrange equations

$$
C_D M \frac{\mathrm{d}C_C}{\mathrm{d}M} + C_C \left(C_D + M \frac{\partial C_D}{\partial M} - 2C_L \frac{\partial C_D}{\partial C_L} \right) = 0
$$

$$
C_D - C_L \frac{\partial C_D}{\partial C_L} = 0
$$

Solution:

$$
M = const = M^*
$$

$$
C_L = const = C_L^*
$$

Optimal control laws:

$$
M = M^*
$$

$$
\delta^*(W) = \frac{W}{q_0 (M^*)^2 C_L^*}
$$

Optimal procedure: **cruise climb at constant M**

Maximum range

$$R_{max} = \int_{W_f}^{W_i} S_R\left(M^*(W), \delta^*(W), W\right)\,\mathrm{d}W$$

$$R_{max} = \frac{L_H}{g}\frac{M^* E^*}{C_C^*}\ln\frac{W_i}{W_f}$$

where $E^* = \dfrac{C_L^*}{C_D(M^*, C_L^*)}$, $C_C^* = C_C(M^*)$

The **Breguet range equation** represents optimal performance

(provided that the constant values of Mach number and lift coefficient are the optimum values M^* and C_L^* defined by the Euler-Lagrange equations)

Symmetric drag polar

$$C_D = C_{D_0}(M) + C_{D_2}(M)C_L^2$$

E-L equations

$$\frac{2}{C_C}\frac{\mathrm{d}C_C}{\mathrm{d}M} + \frac{1}{C_{D_0}}\frac{\mathrm{d}C_{D_0}}{\mathrm{d}M} + \frac{1}{C_{D_2}}\frac{\mathrm{d}C_{D_2}}{\mathrm{d}M} - \frac{2}{M} = 0 \qquad \Longrightarrow \qquad M^*$$

$$C_{D_0} - C_{D_2}C_L^2 = 0 \qquad \Longrightarrow \qquad C_L^* = \sqrt{\frac{C_{D_0}(M^*)}{C_{D_2}(M^*)}}$$

Maximum range

$$R_{max} = \frac{L_H}{g}\frac{M^* E_{max}(M^*)}{C_C(M^*)}\ln\frac{W_i}{W_f}$$

Standard cruise

Constant M, constant h

Symmetric drag polar

Range

$$R = \frac{L_H}{g} \frac{M E_{max}(M)}{C_C(M)} 2 \arctan \frac{q_0 \delta M^2 C_{L,opt}(M)(W_i - W_f)}{\left[q_0 \delta M^2 C_{L,opt}(M)\right]^2 + W_i W_f}$$

where $C_{L,opt} = \sqrt{\dfrac{C_{D_0}(M)}{C_{D_2}(M)}}$

(Ref: N.X. Vinh, Flight Mechanics of High-Performance Aircraft, 1993)

Best range

Best range: constrained maximum range

$R = R(M, \delta) \Longrightarrow$

$$\frac{\partial R}{\partial M} = 0$$
$$\frac{\partial R}{\partial \delta} = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}M}\left[\frac{ME_{max}(M)}{C_C(M)}\right] = 0 \qquad \Longrightarrow \qquad M_{best}$$

$$W_iW_f - \left[q_0 \delta M^2 C_{L,opt}(M)\right]^2 = 0 \qquad \Longrightarrow \qquad \delta_{best} = \frac{\sqrt{W_iW_f}}{q_0 M_{best}^2 C_{L,opt}(M_{best})}$$

Best range

$$R_{best} = \frac{L_H}{g}\frac{M_{best}E_{max}(M_{best})}{C_C(M_{best})} 2 \arctan \frac{W_i - W_f}{2\sqrt{W_iW_f}}$$

Best range (2)

$M_{best} = M^*$ (given by the same E-L equation)

$$C_{L,opt}(M_{best}) = \sqrt{\frac{C_{D_0}(M_{best})}{C_{D_2}(M_{best})}} = \sqrt{\frac{C_{D_0}(M^*)}{C_{D_2}(M^*)}} = C_L^*$$

$$\begin{aligned} \delta_i^* &= \frac{W_i}{q_0 {M^*}^2 C_L^*} \\ \delta_f^* &= \frac{W_f}{q_0 {M^*}^2 C_L^*} \end{aligned} \qquad \Longrightarrow \qquad \delta_{best} = \sqrt{\delta_i^* \delta_f^*}$$

δ_{best} is the geometric mean of the initial and final optimum values δ_i^* and δ_f^* (the initial and final cruise climb values)

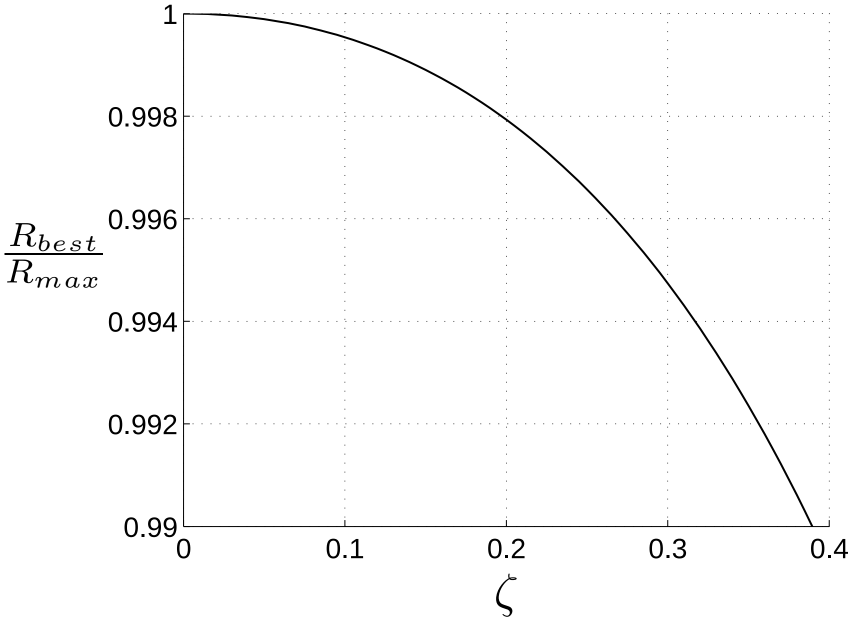
Relationship between the best standard procedure and the optimal procedure: $M_{best} = M^*$ and $\delta_{best} = \sqrt{\delta_i^* \delta_f^*}$

Comparison Best range vs Maximum range

$$\frac{R_{best}}{R_{max}} = \frac{2\arctan\dfrac{W_i - W_f}{2\sqrt{W_i W_f}}}{\ln\dfrac{W_i}{W_f}} = \frac{2\arctan\dfrac{\zeta}{2\sqrt{1-\zeta}}}{\ln\dfrac{1}{1-\zeta}}$$

where $\zeta = \dfrac{W_i - W_f}{W_i} = \dfrac{W_F}{W_i}$

Comparison Best range vs Maximum range (2)



For a fuel load ζ <0.38 one has $\dfrac{R_{best}}{R_{max}}$ >0.9906, that is R_{best} is less than 1 % smaller than R_{max}

Summary

- General aircraft model (high-speed, commercial aircraft)
- Optimal cruise:
  - cruise at constant Mach ($M^*(W) = constant$) and constant C_L
  - cruise climb ($\delta^* = \delta^*(W)$)
- Maximum range: given by the Breguet range equation (optimum performance)
- Best standard cruise quite close to optimal cruise ($M_{best} = M^*$, $\delta_{best} = \sqrt{\delta_i^* \delta_f^*}$)
- Best range: $R_{best} > 0.99 R_{max}$