DEVELOPMENT OF A CLASS OF SHELL FINITE ELEMENTS FOR NONLINEAR APPLICATIONS

R. Winkler
University of Innsbruck, Unit for Engineering Mathematics
Technikerstraße 13, A-6020 Innsbruck
Austria

OVERVIEW

A four node quadrilateral shell element with six global and one internal degrees of freedom (7-parameter model) based on an enhanced assumed strain (EAS) formulation is regarded as a master of a class of shell finite elements particularly applicable for the simulation of complex light weight structures exhibiting severely nonlinear behaviour. Concepts from the recent literature have been adapted and refined in view of challenging industrial applications. The formulation includes thickness change and allows for large rotations and large strains. The element can be employed within commercial codes like ABAQUS, in contrast to alternative formulations using seven global densities of freedom. The performance of the element is comparable to that of solid (or continuum) shell elements. In contrast to the latter, existing FE models (with one node level) can be applied without any change.

1. INTRODUCTION

The present article presents the status quo of one branch of a research project in aerospace engineering, called ICONA (Innovative Concepts for Nonlinear Analysis of Light Weight Structures). The project partners are INTALES Gmbh – Engineering Solutions (Innsbruck) as well as three departments of the University of Innsbruck (Civil Engineering, Mathematics, Computer Science). The project is supported by EADS Astrium ST (Les Mureaux) and TransIT (Innsbruck). It involves a cooperation with Delft University of Technology (Aerospace Engineering). The research is targeted at an improved treatment of complex light weight structures in view of stability analysis, reliability, and robustness. Besides element technology it covers the development of concepts of sensitivity analysis [1,2], numerical methods (domain decomposition, determination of bifurcation and limit points, branching analysis) and databases to handle input and output parameters for the simulation of various and complex load scenarios.

The project is motivated, amongst others, from experiences with the analysis of the frontskirt of ARIANE 5 launcher [3]. One of the main problems in this connexion is the exact determination of bifurcation and limit points. To overcome this problem in the context of Finite Element Method (FEM) it is necessary to operate at element as well as at solver level. The present work is at an early stage and covers the search for adequate element routines containing the required links for an advanced bifurcation and branching analysis (e.g. asymptotic numerical method [4]).

Modern developments in shell element technology tend towards solid shell elements. These elements are treated like solid elements but are optimized for the simulation of shell structures. They are available in commercial codes like ABAQUS[1]. Their two major advantages are:

1) They can be easily coupled to solid elements (discretization of shell intersections, e.g.)
2) Their formulation is directly derived from 3d continuum mechanics and is not restricted to the assumptions of conventional shell theories (plane stress, e.g.)

Unfortunately, they cannot be used within conventional FE models consisting of a single node layer. This fact disagrees with some requirements of the industry, who claim the application of approved models. In order to circumvent this conflict, we search for elements which approach the behaviour of solid elements (allowing for thickness change and large strains) on the one hand, but can be used like conventional shell elements within the existing FE models on the other hand. Elements that satisfy these requirements will be addressed as 3d shell elements. Appropriate formulations have been proposed by several authors [5-13] and are still subject to ongoing research activities [14,15].

One important benefit of 3d (and solid) shell elements is the possibility to implement arbitrary three-dimensional constitutive laws. This involves obvious advantages in view of composite and layered structures [16,17] as well as anisotropic damage. Moreover, similar elements have been used successfully in fields of contact [15] and solid-fluid interaction [18]. The consideration of thickness change suggests describing the shell kinematics by vector components [6,7,8,9,11]. This approach simplifies the implementation compared to conventional formulations using trigonometric functions of rotation angles. In course of the linearization of the element equations no assumptions have to be made concerning small increments in rotational angles. Thus the implementation allows for much larger load increments compared to conventional elements [8]. The latter fact is expected, at least, to compensate the increased computational effort arising from an additional degree of freedom (dof).

In a first step a geometrically non-linear element formulation has been adopted from the literature [11]. It is based on a first-order[2] approximation of the displacement vector (including thickness change) within a continuum mechanics based approach. In order to represent bending deformations correctly, shifting of the midsurface relatively to the top and the bottom surfaces is allowed via an enhanced strain field substituting the missing term of the curvatures. This yields the simplest formulation satisfying the requirements for 3D shell elements (7-parameter model). So far a four node quadrilateral version in updated lagrangian formulation has been implemented within ABAQUS. The implementation is new in that sense that it is totally based on the EAS method. I.e. all locking effects are eliminated by enhanced strain fields. The underlying concepts are partially adopted from the literature: [10,11] (shear and membrane locking and [19] (transverse shear locking). The treatment of curvature thickness (or pinching) locking [20,21] via EAS method is currently under investigation. At the present state an isotropic linear elastic constitutive law is used.

[1] Called continuum elements

[2] With respect to the thickness variable

2. SHELL MODELS

The kinematics of a shell structure can be described by a certain approximation of the kinematic equations of the three-dimensional continuum. Shell elements based on this approach are exactly equivalent to those obtained by degeneration of solid elements (if the same model assumptions are used). In the first case the continuum is reduced to the shell structure first and discretised after. In the latter case the continuum is discretised first and reduced after. Both methods may be addressed as continuum mechanics based or degenerated solid approach. The main disadvantage appears within the consideration of large rotations. To obtain linearised incremental equations the assumption of small increments in rotation angles is needed. This problem is avoided by an alternative approach (geometrically exact or resultant based) which starts from an exact kinematic description of a Cosserat surface and does not contain any restrictions concerning the size of rotations. The formulation is based *a priori* on a postulation of stress resultants in contrast to the definition of stress resultants via thickness integration of the three-dimensional stress tensor. For more detailed information read [22] and references therein.

The presented formulation uses the continuum mechanics based approach. The use of rotation angles is avoided in view of a formulation involving thickness change. The elements based on this formulation are designed for large rotations and large strains. We arrange the displacement of an arbitrary material point of the shell body as a first order approximation with respect to the thickness coordinate:

(1) $\mathbf{u}(\xi^1,\xi^2,\xi^3) = \mathbf{u}^{(0)}(\xi^1,\xi^2) + \xi^3 \mathbf{u}^{(1)}(\xi^1,\xi^2)$

ξ^α are the natural coordinates of the shell midsurface with the corresponding base vectors $\mathbf{A}_\alpha$.[3] The linear dependence with respect to ξ^3 counts for the assumption, that a straight line of material points which is orthogonal to the midsurface in the undeformed configuration remains straight (but not necessarily orthogonal) during deformation. $\mathbf{u}^{(0)}$ describes the deformation of the midsurface and $\mathbf{u}^{(1)}$ the change of the direction and the length of the shell director

(2) $\mathbf{A}_3 = \frac{t}{2} \frac{\mathbf{A}_1 \times \mathbf{A}_2}{|\mathbf{A}_1 \times \mathbf{A}_2|}$

t is the shell thickness. Without any additional kinematic assumption the corresponding shell model is called a 6-parameter model, due to the six independent kinematic dofs. The associated *Green-Lagrange* strains read

(3) $E_{jk} = \varepsilon_{jk} + \xi^3 \chi^{(0)}_{jk} + (\xi^3)^2 \chi^{(1)}_{jk}$

with with the midsurface (membrane) strains $\varepsilon_{\alpha\beta}$, the transverse shear strains $\gamma_\alpha = 2\varepsilon_{\alpha 3}$, the transverse normal strain ε_{33}, the midsurface curvatures $\chi_{\alpha\beta}$ and the "transverse shear curvatures" $\chi^{(0)}_{\alpha 3}$. All other components are zero, in particularly χ_{33} which disagrees with the fact, that the transverse stress varies at least linearly with respect to ξ^3. This unbalance between transverse strain and stress cannot be justified physically and is the reason why the 6-parameter model cannot represent a bending configuration correctly. This problem can be circumvented by either reducing or expanding the ansatz (1).

The reduced ansatz with fife kinematic dofs (5-parameter model)

(4) $\mathbf{u} = \mathbf{u}^{(0)} + \xi^3 \, \vartheta_\alpha \, \mathbf{A}^\alpha$

conserves the length of the shell director (except for a second order term with respect to the rotation angles ϑ_1 and ϑ_2) and thus corresponds to the *Reissner-Mindlin* kinematic assumption: a straight line of material points which is orthogonal to the midsurface in the undeformed configuration remains straight and unstretched during deformation [22]. Therefore the transverse strain E_{33} can be eliminated from the formulation via the plain stress assumption ($S^{33} = 0$) which leads to a modified constitutive law. The remaining strain components are $\varepsilon_{\alpha\beta}$ and γ_α.

The extended ansatz[4]

(5) $\mathbf{u} = \mathbf{u}^{(0)} + \xi^3 \mathbf{u}^{(1)} + (\xi^3)^2 q \, \mathbf{a}_3$

leads to a 7-parameter model [9,13,14]. Now E_{33} varies quadratically with respect to ξ^3 which is consistent with the characteristic of S_{33} except for third order terms. The transverse normal strain $\varepsilon_{33} = (\tau^2 - 1)/2$ is governed by the relative thickness change $\tau = |\mathbf{a}_3| / |\mathbf{A}_3|$. The "transverse curvature" $\chi_{33} = 4\tau^2 q \, (1 + \xi^3 q)$ may be interpreted as a displacement of the midsurface relatively to the bottom and top surfaces of the shell body. The main disadvantage of this model is the fact that every material point of the midsurface is provided with seven kinematic dofs. This causes an increased computational effort for FE calculations based on that model compared to classical shell models with 3 (*Kirchhoff-Love*) or 5 (*Reissner-Mindlin*) dofs. Only in the case of a linear constitutive law, the seventh dof may be eliminated by the assumption that its work conjugate counterpart vanishes [9].

A generally applicable method to avoid the seventh global dof was described first by Büchter and Ramm [6]. They start from a 6-parameter formulation and introduce an additional strain variable $\tilde{E}_{33}$ via enhanced assumed strain (EAS) method in order to eliminate the unbalance of E_{33} and S_{33}. This idea has been developed further by a few authors [10,11] but ongoing research activities are missing since [12,23].[5] It is considered the most promising approach view of many large scale applications and therefore the origin of the presented formulation (chapter 3).

[3] Greek indices count from 1 to 2, small Latin indices from 1 to 3. Summation convention is used.

[4] $\mathbf{a}_3 = \mathbf{A}_3 + \mathbf{u}^{(1)}$

[5] The most recent results in the field of 7-parameter elements [14,15] deal with 7 global dofs

3. FINITE ELEMENT DISCRETIZATION

Originally the EAS method has been developed by Simo *et al.* to eliminate shear and membrane locking in plate elements by enhancing the membrane strains [24,25]. The extension to the non-linear case has been performed by Andelfinger and Ramm [26]. By further developments the method is adequate to treat all locking effects [27]. The method is based on the variational principle

$$\text{(6)}\quad \int_{V_e}\left(\tilde{S}^{ij}\cdot\delta\tilde{E}_{ij}-\overset{u}{S}^{ij}\cdot\delta\overset{u}{E}_{ij}\right)dV-\delta\Pi_{\text{ext}}=0$$

which is derived from the modified Veubeke-Hu-Washizu principle [28] by splitting the strains

$$\text{(7)}\quad E_{ij}=\overset{u}{E}_{ij}+\tilde{E}_{ij}$$

into displacement compatible strains $\overset{u}{E}_{ij}$ and enhanced strains $\tilde{E}_{ij}$ and postulating the orthogonality constraint [24]

$$\text{(8)}\quad \int_V S^{ij}\tilde{E}_{ij}\,d^3x=0$$

Both variables ($\mathbf{u}$ and $\tilde{\mathbf{E}}$) have to be discretized:[6]

$$\text{(9)}\quad \begin{aligned}&\mathbf{u}^h=\mathcal{N}\,\underline{d}\quad\Rightarrow\quad \overset{u}{\underline{E}}{}^h=\mathcal{B}\,\underline{d}\\ &\underline{\tilde{E}}^h=\mathcal{M}\,\underline{\alpha}\end{aligned}$$

The nodal displacement vector $\underline{d}$ contains the kinematic dofs of each node of the element and the Matrix $\mathcal{N}$ the element shape functions. $\mathcal{B}$ is the strain-displacement matrix. In the context of FE formulation strains and stresses are treated as matrices as well.[7] The ansatz functions $\mathcal{M}_{KL}\in L^2$ are not uniquely determined by the shape of the element. They have to be chosen according to the requirements for the enhanced strains (chapter 4). With respect to the variational index, the enhanced strains need not to be continuous between the elements. Therefore the coefficients α_K are determined at element level and need not contribute to the global DOFs.

Linearization yields the incremental linear system of equations

$$\text{(10)}\quad {}^{n}\begin{bmatrix}\mathcal{K}_{uu}+\mathcal{K}_G & \mathcal{K}_{u\alpha}\\ \mathcal{K}_{\alpha u} & \mathcal{K}_{\alpha\alpha}\end{bmatrix}\begin{bmatrix}\Delta\underline{d}\\ \Delta\underline{\alpha}\end{bmatrix}={}^{n+1}\begin{bmatrix}\underline{F}_{\text{ext}}\\ \underline{0}\end{bmatrix}-{}^{n}\begin{bmatrix}\underline{F}_{\text{int}}\\ \underline{\tilde{F}}_{\text{int}}\end{bmatrix}$$

with the stiffness matices

$$\text{(11)}\quad \mathcal{K}_{uu}=\int_{V_e}\mathcal{B}^T C\,\mathcal{B}\,d^3x\,,\qquad (\mathcal{K}_G)_{IJ}=\int_{V_e}\overset{u}{S}{}^h_K\frac{\partial\mathcal{B}_{KI}}{\partial d_J}\,d^3x$$

$$\text{(12)}\quad \mathcal{K}_{u\alpha}=\int_{V_e}\mathcal{M}^T C\,\mathcal{B}\,d^3x\,,\qquad \mathcal{K}_{\alpha u}=\int_{V_e}\mathcal{B}^T C\,\mathcal{M}\,d^3x$$

and the forces

$$\text{(13)}\quad \underline{F}_{\text{int}}=\int_{V_e}\mathcal{B}\,\overset{u}{\underline{S}}{}^h\,d^3x\,,\qquad \underline{\tilde{F}}_{\text{int}}=\int_{V_e}\mathcal{M}\,\underline{\tilde{S}}^h\,d^3x$$

$$\text{(14)}\quad \underline{F}^T_{\text{ext}}=\frac{\partial\Pi^h_{\text{ext}}}{\partial\underline{d}}$$

$\Delta\underline{d}=\Delta d^{(1)}$ and $\Delta\underline{\alpha}=\Delta\alpha^{(1)}$ are first estimations of ${}^{n+1}_{\;\;n}\Delta d$ and ${}^{n+1}_{\;\;n}\Delta\alpha$. The correct values are found by Newton iteration:[8]

$$\text{(15)}\quad {}^{n+1,i-1}\begin{bmatrix}\mathcal{K}_{uu}+\mathcal{K}_G & \mathcal{K}_{u\alpha}\\ \mathcal{K}_{\alpha u} & \mathcal{K}_{\alpha\alpha}\end{bmatrix}\begin{bmatrix}\Delta\underline{d}^{(i)}\\ \Delta\underline{\alpha}^{(i)}\end{bmatrix}={}^{n+1}\begin{bmatrix}\underline{F}_{\text{ext}}\\ \underline{0}\end{bmatrix}-{}^{n+1,i-1}\begin{bmatrix}\underline{F}_{\text{int}}\\ \underline{\tilde{F}}_{\text{int}}\end{bmatrix}$$

$$\text{(16)}\quad {}^{n+1,i}_{\;\;\;n}\Delta\underline{d}=\sum_{j=1}^{i}\Delta\underline{d}^{(j)}\quad\text{and}\quad {}^{n+1,i}_{\;\;\;n}\Delta\underline{\alpha}=\sum_{j=1}^{i}\Delta\underline{\alpha}^{(j)}$$

As stated above, the strain parameters $\underline{\alpha}$ may be eliminated at element level:

$$\text{(17)}\quad \Delta\underline{\alpha}^{(i)}=\left({}^{n+1,i-1}\mathcal{K}_{\alpha\alpha}\right)^{-1}\cdot\left({}^{n+1,i-1}\underline{\tilde{F}}_{\text{int}}-{}^{n+1,i-1}\mathcal{K}_{\alpha u}\,\Delta\underline{d}^{(i)}\right)$$

Thus the incremental system of equations takes the same shape than in the case of a purely displacement based formulation:

$$\text{(18)}\quad {}^{n+1,i-1}\hat{\mathcal{K}}\,\Delta\underline{d}^{(i)}={}^{n+1}\underline{F}_{\text{ext}}-{}^{n+1,i-1}\underline{\hat{F}}_{\text{int}}$$

with a “softened” stiffness matrix

$$\text{(19)}\quad \hat{\mathcal{K}}=\mathcal{K}_{uu}+\mathcal{K}_G-\mathcal{K}_{u\alpha}\mathcal{K}_{\alpha\alpha}^{-1}\mathcal{K}_{\alpha u}$$

and a modified internal load vector

$$\text{(20)}\quad \underline{\hat{F}}_{\text{int}}=\underline{F}_{\text{int}}-\mathcal{K}_{u\alpha}\mathcal{K}_{\alpha\alpha}^{-1}\underline{\tilde{F}}_{\text{int}}$$

[6] Discretized quantities are indicated by a superscript h, which refers to the characteristic element size

[7] $\underline{E}=(E_{11}\quad E_{22}\quad E_{33}\quad 2E_{12}\quad 2E_{13}\quad 2E_{23})^T$
$\underline{S}=(S_{11}\quad S_{22}\quad S_{33}\quad S_{12}\quad S_{13}\quad S_{23})^T$

[8] n indicates the last converged load increment. i labels the iteration step of the current increment $n+1$

4. ENHANCED STRAIN FIELDS

In order to gain a totally EAS based 7-parameter element, the enhanced strain field $\tilde{E}_{ij} = \tilde{\varepsilon}_{ij} + \xi^3 \tilde{\chi}_{ij}$ hast to include all zero order components $\tilde{\varepsilon}_{ij}$ and the first order component $\tilde{\chi}_{33}$. The zero order components compensate parasitic strain due to locking phenomena: shear and membrane locking ($\tilde{\varepsilon}_{\alpha\beta}$), transverse shear locking ($\tilde{\varepsilon}_{\alpha 3}$), curvature thickness locking ($\tilde{\varepsilon}_{33}$). The first order component induces the missing midsurface shift for bending deformations. The corresponding ansatz functions are contained in the matrix

(21) $$\mathcal{M} = \begin{bmatrix} \mathcal{N}^{(\alpha\beta)} & \mathcal{N}^{(\alpha 3)} & \mathcal{N}^{(33)} & 0 \\ 0 & 0 & 0 & \mathcal{M}^{(33)} \end{bmatrix}$$

In detail, for transverse curvature (midsurface shift) [6]

(22) $$\mathcal{M}^{(33)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & \xi^1 & \xi^2 & \xi^1\xi^2 \end{bmatrix}$$

for the membrane strains (shear and membrane locking) [10]

(23) $$\mathcal{N}^{(\alpha\beta)} = \begin{bmatrix} \xi^1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \xi^1 & \xi^2 & \xi^1\xi^2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \xi^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

for the transverse shear strains (transverse shear locking) [27]

(24) $$\mathcal{N}^{(\alpha 3)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ N_{,2} & 0 \\ 0 & 0 \\ 0 & N_{,1} \\ 0 & 0 \end{bmatrix}, \qquad N = \left[1 - (\xi^1)^2\right]^{\frac{1}{2}} \cdot \left[1 - (\xi^2)^2\right]^{\frac{1}{2}}$$

and for transverse normal strains (curvature thickness locking)

(25) $$\mathcal{N}^{(33)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \xi^1 & \xi^2 & \xi^1\xi^2 \end{bmatrix}$$

5. OUTLOOK

The presented element layout is the origin of a sequence of element formulations which are considered to approach industrial applicability successively. The most important ones of the subsequent steps are the incorporation of Mises plasticity, followed by thermo-mechanical coupling. The treatment of shell structures with kinks and/or intersections is an open item of great practical importance. Eight and nine node derivatives may be required in view of cross-checking analyses. Special numerical efficiency is expected from a one-point integrated implementation.

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