OPTIMAL MANOEUVRE CHANGE-DETECTION OF AGILE AERIAL SYSTEMS

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OVERVIEW

We are considering the problem of tracking the motion of a flying object which is capable of performing unpredictable and abrupt manoeuvre changes. The motion of such a system can be modelled by a random jump process (U_t) which can be interpreted as the state of a discrete state machine determining the mode (or kind) of manoeuvre.

The motion X_t of the vehicle can be modelled by a, in general, vector-valued stochastic differential equation

$$dX_t = a(X_t, U_t)dt + \sigma(X_t, U_t)\, dW_t$$

This equation describes the system which may consist of a guidance law and/or aerodynamics. These in turn provide deterministic accelerations or kinematics which may be described in the system equation shown above.

The physical motion shall be measured by an appropriate sensor with measurements (Y_t), again modelled by a very general stochastic differential equation

$$dY_t = b(X_t, Y_t)dt + \nu(Y_t)\, dV_t$$

This presentation discusses the problem of estimating (U_t, X_t) given all measurements $(Y_s)_{s\le t}$. An optimal estimation formula is derived. It turns out that for relevant special cases the formula is closed and can be directly calculated. Simulation results are presented.

The used method of estimation generates an optimal estimate with can be proven to be optimal in the least variance sense. A similar estimate has already been successfully applied to a problem in the field of terrain reference navigation (see [3]). On the other hand, the investigated estimation process requires huge effort. There are more heuristic approaches which deal with similar problems and produce good (but nevertheless suboptimal) solutions, see e. g. [2].

1. CONVENTIONS, BASIC SETUP

Let $(\Omega, \mathfrak{F}, P, (\mathfrak{F}_t))$ be a filtered probability space. By (W_t) and (V_t) we denote $(\mathfrak{F}_t)$-adapted, independent vector-valued *Brownian motions* with the suitable amount of dimensions.. Let (X_t) be a stochastic process. By $\mathfrak{F}^X$ we denote the *generated filtration* (i. e., the coarsest filtration with respect to which (X_t) is adapted). A *martingale* (M_t) with respect to a filtration $(\mathfrak{F}_t)$ is a stochastic process which holds $E(M_t \mid \mathfrak{F}_s) = M_s$ for all $s \le t$. A stochastic process (X_t) is a *semimartingale* if it allows the representation

(1) $X_t = X_0 + A_t + M_t$

with an $\mathfrak{F}_0$-measurable random variable X_0 with existing variance, an $(\mathfrak{F}_t)$-adapted predictable process (A_t) and an $(\mathfrak{F}_t)$-adapted martingale (M_t). A semimartingale (X_t) is said to be a *smooth semimartingale*, if the predictable component is differentiable (to be precise, absolute continuity of (A_t) would be sufficient),

(2) $A_t = \int_0^t a_s\, ds$.

2. THE VEHICLE'S MOTION

The motion X_t of an aerial vehicle can be modelled by a general stochastic differential equation

(3) $dX_t = a(X_t, U_t)dt + \sigma(X_t, U_t)\, dW_t$

Typically, U_t determines the currently applied guidance law which may instantaneously change. In the simplest case constant accelerations of a one-dimensional motion are commanded and integrated to velocities and positions,

(4) $X_t = \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix},\ a(X_t, U_t) = \begin{pmatrix} X_2(t) \\ a(U_t) \end{pmatrix}.$

It turns out that typical jump processes, such as continuous time Markov chains, are well suited to describe the random switching of the guidance.

3. THE UNDERLYING DISCRETE STATE PROCESS

The motion of such a system can be modelled by a random jump process (U_t) which may be interpreted as the state of a discrete state machine determining the mode (or kind) of manoeuvre.

Let $\mathbf{U}$ be the state set, $U_t \in \mathbf{U}$, and $\pi_k := P(U_0 = k)$ be the initial distribution of $(U_t)_{t \geq T_0}$ for all $k \in \mathbf{U}$. By (T_n) we denote the jump times of the process, $T_0 = 0$.

A process is called Markovian, if it holds $E(U_t \mid \mathfrak{F}_s) = E(U_t \mid U_s)$ for all $t \geq s$. (U_t) is defined to be stationary, which means that its intensities

(5) $\lambda_{ik}(t) := \lim_{h \to 0} \frac{1}{h} \left(P(U_{t+h} = k \mid U_t = i) \right)$

($i, k \in \mathbf{U}$, $i \neq k$) are independent of time and may therefore be written as a constant matrix $\Lambda = (\lambda_{ik})$, where by convention

(6) $\lambda_{kk} := 0$.

The so-called interarrival times $\Delta T_n := T_n - T_{n-1}$ of a stationary Markov process with discrete states are independent of each other and exponentially distributed,

(7) $P(\Delta T_n \leq s, U_{T_n} = k \mid U_{T_{n-1}} = i) = 1 - e^{-\lambda_{ki} t}$

4. THE MEASUREMENT PROCESS

The physical motion shall be measured by an appropriate sensor with measurements (Y_t), modelled by a very general stochastic differential equation

(8) $dY_t = b(X_t, Y_t)\, dt + \nu(Y_t)\, dV_t$.

Observe that three different kinds of stochastic effects influence the measurement: The random mode (U_t) of the manoeuvre, the random system noise (W_t), which affects the motion such as wind or parameter uncertainties, and the measurement noise (V_t), which results in an imperfect measurement of the random movement of the object.

5. THE FILTERING PROBLEM

The considered problem is described as follows: Estimate the value of (U_t, X_t) given all observations $(Y_s)_{s \leq t}$. The estimation shall be optimal in the minimal variance sense. It can be proven that this problem is equivalent to the calculation of the conditional expectation

$$E((U_t, X_t) \mid \mathfrak{F}_t^Y)$$

In the classical linear case with normally distributed random variables this conditional expectation can be calculated just by using conditional expectations of the first two moments

$$E((U_s, X_s) \mid \mathfrak{F}_s^Y),\; E((U_t, X_t)(U_t, X_t)^T \mid \mathfrak{F}_t^Y)$$

with respect to past times $s \leq t$. After some simplification, the resulting formulas describe the classical Kalman-Bucy-Filter.

In the non-linear case, the problem becomes tremendously more complicated since the model may generate arbitrary distributions which cannot be reconstructed from the moments.

A key point to cope with this situation is to choose an appropriate function $f(U, X, t)$ which generates an $(\mathfrak{F}_t)$-adapted smooth semimartingale $R_t = f(U, X, t)$, and to analyse the properties of such a process.

For a smooth semimartingale (R_t) with representation (1) it is a topic of the non-linear filtering theory (see [1]) to determine the appropriate projected representation (i. e. conditioned with respect to the σ -Algebra $\mathfrak{F}_t^Y$ generated by all measurements up to time t). Let $\tilde{R}_t := E(R_t \mid \mathfrak{F}_t^Y)$, etc. A relatively straight forward calculation (see e.g. [1, chapter 8] for the semimartingale calculus in the context of non-linear estimation) produces the result

(9) $$d\tilde{R}_t = \tilde{a}(X_t, U_t)\, dt + \left(\frac{dE\left(\langle \tilde{M}, W \rangle_t \mid \mathfrak{F}_t^Y \right)}{dt} + \frac{E\left(R_t b(X_t, Y_t) \mid \mathfrak{F}_t^Y \right) - \tilde{R}_t \tilde{b}(X_t, Y_t)}{\sigma_t(Y_t)} \right) d\tilde{W}_t$$

with a Brownian motion

$$d\tilde{W}_t = \sigma_t^{-1} \left(dY_t - \tilde{b}(X_t, Y_t)\, dt \right).$$

Since the martingale (M_t) and the system noise (W_t) are stochastically independent, the same applies to the projected martingale and the system noise. Hence, its quadratic covariation process disappears.

One of the major problems in non-linear filtering is that the resulting formulas may require estimates of variables which are even more complex to derive. In our case, the estimation process of R_t *requires* estimates of $a(X_s, U_s)$, X_s and $R_s b(X_s, Y_s)$ for all $s \leq t$ including of course, the past estimates of R_s itself. It is of course possible to restate a similar filter problem for these required estimates. We would get new filtering equations which again would require further estimates and so on. To break this vicious circle it is necessary to find a suitable class of functionals f which lead to a closed estimation process.

It can be easily shown that if we do not wish to confine ourselves to trivial cases for a, it is necessary to estimate the complete distribution of (X_t, U_t).

The semimartingale representation for a process (R_t) of the form

(10) $R_t = f(X_t, U_t)$

with an arbitrary function f being twice continuously differentiable can be calculated as

(11) $$df(X_t,U_t) = \left(a(X,U_t) f_1 + \frac{\sigma(X_t,U_t)^2}{2} f_{11} + f_2 + r \right) dt + dM_t$$

where f_1 etc. denotes the derivatives of f at (X_t, U_t) and r a functional depending on f, X_t, U_t and on the transition intensities (λ_{ik}).

Substitution of the predictable component into the general filtering formula yields (with $E_t(\cdot) := E(\cdot \mid \mathfrak{F}_t^Y)$)

(12) $$\begin{aligned} d_t E_t\left(f(X_t,U_t)\right) &= E_t\left(a(X_t,U_t) f_1 + \tfrac{1}{2}\sigma(t,X_t)^2 f_{11} + f_2 + r\right) dt \\ &+ E_t\left(f(X_t,U_t) A(X_t,Y_t)\right) - \left(E_t(f(X_t,U_t)) E_t(A(X_t,Y_t)\right) \frac{d\tilde{W}_t}{\upsilon(Y_t)} \end{aligned}$$

Let ρ_t be the common conditional "density" function,

(13) $$\rho_t(x,u) := \frac{\partial P(X_t \le x, U_t = u \mid \mathfrak{F}_t^Y)}{\partial y},$$

(the existence of the conditional density is assumed in the sequel). Taking into account

(14) $$E_t(f(X_t,U_t) \mid \mathfrak{F}_t^Y) = \sum_u \int_{x=-\infty}^{\infty} f(x,u) \rho_t(x,u) \, dx$$

one finds after integration by parts and rearranging terms the resulting filtering formula

(15) $$\begin{aligned} d_t \rho_t(x,u) &= \left[-a(x,u) \frac{\partial \rho(x,u)}{\partial x} + \frac{1}{2} \sigma(x,u)^2 \frac{\partial^2 \rho(x,u)}{\partial x^2} + \overline{r} \right] dt \\ &+ \left[A(x,Y_t)\rho(x,u) - E_t(A(X_t,Y_t))\rho(x,u) \right] \frac{d\tilde{W}_t}{\upsilon(Y_t)} \end{aligned}$$

Hereby, $\overline{r}$ is a functional only depending on ρ_t and on the transition intensities (λ_{ik}).

The filtering formula is a partial differential equation with three distinct components on the right hand side which can be interpreted separately. The first part consists of a generalized transport equation (with a diffusion component)

(16) $$d_t \rho_t(x,u) = \left[-a(x,u) \frac{\partial \rho(x,u)}{\partial x} + \frac{1}{2} \sigma(x,u)^2 \frac{\partial^2 \rho(x,u)}{\partial x^2} \right] dt$$

which, in absence of other effects, shifts the velocity density according to the modelled accelerations a und broadens it due to the system noise defined by σ^2. The second term $\overline{r}\,dt$ takes into account the jump intensities λ of our modelled process. Density values $\rho_t(x,u)$ for states with positive transition balance[1] are increased; density values for states with negative transition balance are diminished. While the first two terms refer to the system model, the third term handles new measurement increments dY_t, which are translated to innovations $d\tilde{W}_t$,

(17) $$d\tilde{W}_t = \sigma_t^{-1} \left(dY_t - \tilde{b}(X_t, Y_t) dt \right).$$

Pretty similar to linear filters, the innovations are weighted by the covariance-like quantity

(18) $$A(x,Y_t)\rho_t(x,u) - E_t(A(X_t,Y_t))\rho_t(x,u)$$

with

(19) $$E_t(A(X_t,Y_t)) = \sum_u \int_{x=-\infty}^{\infty} A(x,Y_t) \rho_t(x,u) \, dx.$$

6. AN EXAMPLE

We assume that three different manoeuvres with constant accelerations may occur with random switching times according to a Markov process (U_t) as described above. The accelerations are defined to be $10\frac{m}{s^2}$, $0\frac{m}{s^2}$, $-10\frac{m}{s^2}$ for $U_t = 1,2,3$, respectively, with a random (system) of $\sigma_t^2 = 0.1^2 \frac{m^2}{s^5}$. The realised position is continuously measured, the measurements are degraded by a measurement noise of $\upsilon_t^2 = 1\frac{m^2}{s^3}$. For simplicity, let the transition intensities be $\lambda_{12} = \lambda_{13} = 1.5$ and $\lambda_{jk} = 0$ otherwise. The starting distributions are given by $\pi(1) = P(U_0 = 1) = 1$, $X_0 = 0m$, $V_0 \sim N(0\frac{m}{s}, 5^2\frac{m^2}{s^2})$.

In the following, the simulation results of one simulation run are depicted.

[1] A state is said to have a positive transition balance, iff the probability that another state k performs a transition to u within the considered time span dt is greater than the probability that a transition out of state u happens

Figure 1 shows that the simulated manoeuvre 1 (acceleration of (noisy) $10\frac{m}{s^2}$) changes after 0.45s to manoeuvre 3 (acceleration of (noisy) $-10\frac{m}{s^2}$).

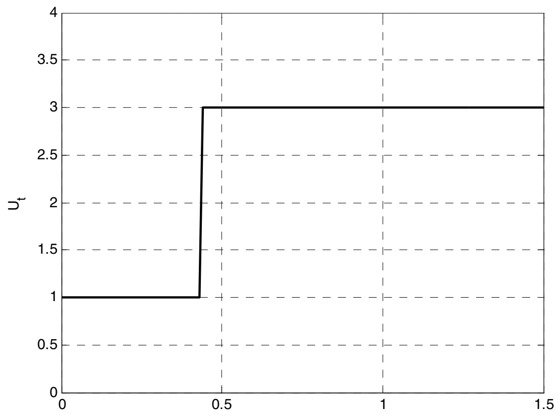

FIG 1. The manoeuvre selection process (U_t)

In figure 2 the simulated measurement is depicted. For clarification the numerically differentiated measurements (representing velocities) are shown in figure 3. Since the system noise is chosen to be quite moderate relative to the measurement noise, an accurate measurement should show an almost linear increase of velocity for the first 0.45s and then an almost linear decrease.

FIG 2. The simulated measurements (position) (Y_t)

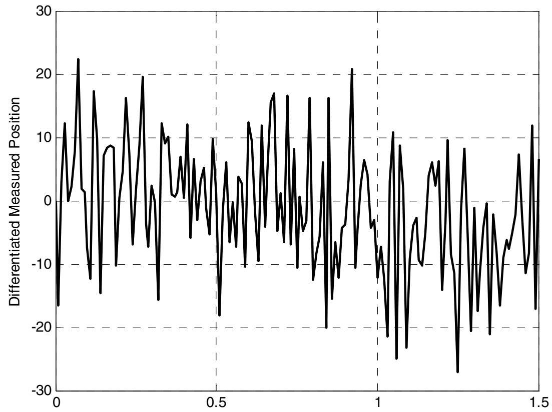

FIG 3. The numerically differentiated measurements (velocity)

The filtered common density ρ_t for time $t = 1.5s$ is depicted in figure 4 (the values for $U_t = 1$ are very small). It can be seen that in this simple example asymptotic normal conditional distributions arise, the expected velocities are expected to be about $10\frac{m}{s}$. In figure 5 the evolution of the projected probabilities

$$P_t(U_t = u) := P(U_t = k \mid \mathfrak{F}_t^Y) = \int_{x=-\infty}^{\infty} \rho_t(x,u)\,dx$$

are shown over time. Due to the poor measurement information, the model assumptions determine the estimation process in the beginning. These assumptions include transitions according to the specified transition rates and the system dynamics according to the constant accelerations. On the long term, the measurements allow the estimation and discrimination of the applied manoeuvres over time.

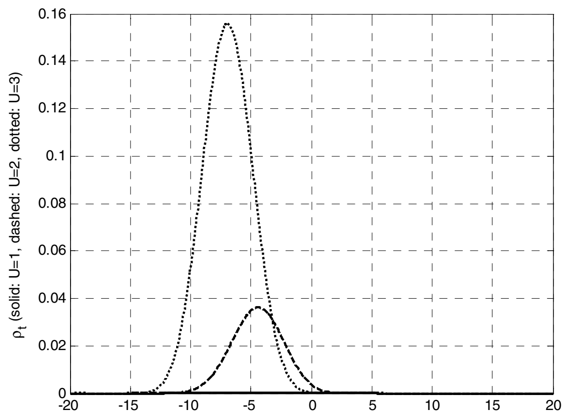

FIG 4. The estimated probability densities $\rho_t(x,u)$ for $t = 1.5s$

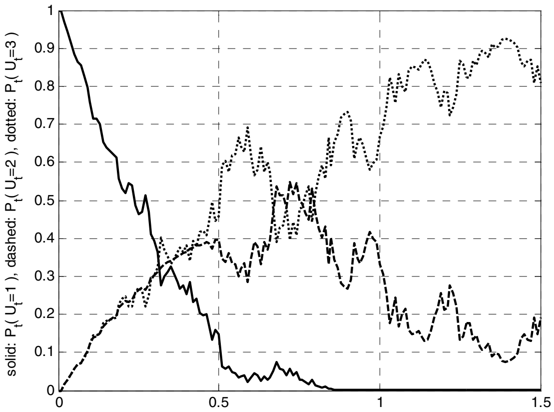

FIG 5. The manoeuvres probabilities given the measurements (Y_t)

Appendix

[1] G. Kallianpur: *Stochastic Filtering Theory*, Springer Verlag, New York, 1980.
[2] S. P. Lindner, C. Schell: *A Non-Bayesian Segmenting Tracker for Highly Manoeuvring Targets*, IEEE Transactions on Aerospace and Electronic Systems, Vol. 41, No 4, pp. 1168-1177, 2005
[3] W. Engelhardt, A. Schöttl: *Reference Model Errors in Multisensor Navigation Systems*, proceedings of the Fifth International Military Sensing Symposium, Gaithersburg, MD, USA, 2002.