AIRCRAFTS CONTROL SYSTEMS DESIGN: AN H_∞ LOOP-SHAPING APPROACH

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Abstract

Loop-shaping design with H_∞ controller synthesis has a role of primary importance for the synthesis of robust multivariable controllers. However, a typical difficulty is that the designer has to develop considerable expertise in this matter before obtaining good results. In fact, the choice of the weight matrices, that are at the heart of the loop-shaping design, is not an easy issue.
The aim of this work is to improve the well known H_∞ loop-shaping approach, by simplifying the shaping procedure and, at the same time, to render the controlled plant less sensitive to model uncertainties. The main idea consists in the definition of a scaling method aiming to minimizing the system condition number. This simplifies the choice of the weighting matrices and increases the system robustness.
The proposed procedure makes use of the 2-norm condition number and the relative gain array (RGA) as indicators of the system controllability and of its sensitivity to model uncertainties. This work is validated by simulations with Matlab/Simulink using an aircraft linearized model.

1 Nomenclature

A	=	generic matrix
D	=	diagonal matrix ($D = \mathrm{diag}\{D_I^{-1}, D_O\}$)
D_I	=	input diagonal scaling matrix
D_O	=	output diagonal scaling matrix
G	=	plant transfer function
G_S	=	plant transfer function after shaping
H	=	matrix [see Eq. (6)]
K	=	feedback controller
K_∞	=	feedback H_∞ controller
M	=	Mach number
V	=	flight velocity
W_1	=	overall pre-compensator
W_2	=	overall post-compensator
W_a	=	align matrix
W_g	=	pre-compensator for control of actuator usage
W_p	=	pre-compensator containing dynamic shaping
a_{ij}	=	element (row i, column j) of matrix A
h	=	flight altitude
k	=	constant [see Eq. (10)]
m	=	order of the generic plant
q	=	pitch angular velocity
t	=	time
Λ	=	Relative Gain Array (RGA)
α	=	incidence angle
γ	=	2-norm condition number
$\gamma^\star$	=	optimum 2-norm condition number
δ_c	=	canard deflection
δ_e	=	elevon deflection
ϵ	=	tolerance of the minimization process
θ	=	pitch attitude angle
λ_{ij}	=	element (row i, column j) of RGA
$\overline{\sigma}$	=	maximum singular value
$\underline{\sigma}$	=	minimum singular value
ω_d	=	design frequency

Subscripts

$i1$	=	induced 1-norm
$i\infty$	=	induced ∞-norm
sum	=	sum matrix norm
$\|\|\cdot\|\|_m$	=	$2\max\{\|\|\cdot\|\|_{i1}, \|\|\cdot\|\|_{i\infty}\}$

Superscripts

†	=	pseudo-inverse
⋆	=	optimum scaling

2 Introduction

Among the existing methods for designing robust multivariable control systems via H_∞ optimization, the loop-shaping design is particularly attractive. It uses weighting functions to shape the system open-loop transfer function and the system robustness is maximized against coprime factor uncertainties, instead of additive or multiplicative uncertainties, thus improving the robustness properties of the controlled plant. Other peculiarities of H_∞ design method can be found in Ref. [1].
The loop-shaping design has been developed in 1990 by McFarlane and Glover,[2] and, subsequently, has been the subject of considerable developments and applications in the aerospace field. In essence, the methodology consists in a two stage design process.[3,4] First, a pre- and a post-compensator W_1 and W_2 are designed and applied to the open-loop plant to give the desired shape to the singular values of the open-loop frequency response. The shaped plant is then robustly stabilized with respect to coprime factor uncertainty through H_∞ optimization, which provides

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the controller K_∞. This second step does not require any weight selection. If the robust stabilization is successful, the shape of the singular values of the robustly stabilized plant results to be similar to that of the open-loop shaped plant. The commonly used set-up of the loop-shaping controller is shown in Figure 1.

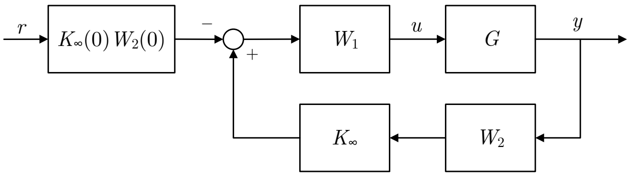

Fig. 1: Typical implementation for a loop-shaping controller.

The classical and well known procedure for H_∞ loop-shaping design has been developed by Hyde.[5] It is a systematic step by step procedure, that we briefly summarize for convenience:

1. Scale all inputs and outputs. In practice the common approach is to make variables less than one in magnitude, by dividing them by their maximum expected or allowed changes.

2. Rearrange the inputs so that the plant is as diagonal as possible.

3. Select the elements of the diagonal pre- and post-compensators so that the singular values of the shaped plant $G_S = W_2\, G\, W_p$ reach the desired shape (typically high gains at low frequencies, a roll-off of about 20dB/decade in the bandwidth and higher slopes at high frequencies). W_p contains the dynamic shaping, while W_2 has constant terms reflecting the relative importance of the measures fed back to the controller (usually W_2 is the identity matrix).

4. Align the singular values at the desired bandwidth by means of a diagonal align matrix W_a, of constant elements, cascaded with W_p . This step has to be carried out only if the plant is not ill-conditioned (see below).

5. An additional diagonal gain matrix W_g can be added, cascaded to W_a, to provide control over actuator usage. After this step the shaped plant will result $G_S = W_2\, G\, W_1$, with $W_1 = W_p\, W_a\, W_g$.

6. Perform a left-coprime factor stabilization on G_S and determine K_∞. The resulting overall controller is $K = W_1 K_\infty W_2$.

7. Create the closed loop and check the system performances in terms of time responses, actuator use and robustness.

8. If necessary, alter W_1 and W_2 and re-run the stabilization process.

The design procedure proposed in this paper modifies that described above in both the scaling of the plant and the definition of the weight functions. The main purpose is to simplify the design of pre- and post-compensators and to reduce the sensitivity of the controlled plant to uncertainties.
The paper is organized as follows: the concepts of 2-norm condition number and the relative gain array are first recalled, together with an overview their useful properties. Then the H_∞ controller design procedure is described in detail, and finally a case study is discussed.

3 Condition Number and Relative Gain Array

Two measures widely used to quantify both the degree of directionality and the level of interactions in MIMO systems are the condition number and the relative gain array (RGA). A definition of both these measures, that are useful to understand the proposed design procedure, is given below along with their main properties. For a more detailed discussion the reader is referred to Refs. [6, 7].

Condition Number: the condition number in 2-norm is defined as the ratio between the maximum and the minimum singular values of the plant, that is

$$\gamma(G) \triangleq \overline{\sigma}(G)/\underline{\sigma}(G) \tag{1}$$

The condition number strongly depends on the scaling of the inputs and outputs. This implies the definition of the *minimized condition number* $\gamma^\star$:

$$\gamma^\star(G) \triangleq \min_{D_I, D_O} \gamma(D_O\, G\, D_I) \tag{2}$$

where D_I and D_O are input and output diagonal scaling matrices. A matrix with large condition number is said to be *ill-conditioned*. Moreover, a large condition number (typically larger than 10[7]) may indicate control problems[8] because:

1. It may be caused by a small value of $\underline{\sigma}(G)$, which is usually undesirable.

2. It may imply a large value of $\gamma^\star$, meaning that the plant has large RGA elements and, therefore, control problems (as described below).

3. It may indicate sensitivity to uncertainties. This is not true in general, but the reverse holds.

4. The matrix required for the alignment of the singular values of the plant at the desired bandwidth will have a large condition number that can yield poor robustness.

Relative Gain Array (RGA): the RGA of a complex non-singular square matrix is a square matrix defined as

$$\Lambda(G) = G \times (G^{-1})^T \tag{3}$$

where $\times$ denotes element-by-element multiplication.
The RGA has many interesting properties, among which the most important for our goals are:

1. It is independent of input and output scaling.
2. The sum-norm of the RGA, $\|\Lambda\|_{\text{sum}}$ defined below, is very colse to $\gamma^\star$: this means that a plant with large RGA-elements is always ill-conditioned.
3. Plants with large RGA-elements around the crossover frequency are affected by high sensitivity to input uncertainty.
4. Large RGA-elements indicate sensitivity to element-by-element uncertainty.

The RGA-number is defined as follows:

$$\text{RGA-number} \triangleq \|\Lambda(G) - I\|_{\text{sum}} \tag{4}$$

where $\|A\|_{\text{sum}} = \sum_{i,j} |a_{ij}|$ is the sum matrix norm. The smaller the RGA-number is, the more diagonally dominant the plant.
The RGA concept may be generalized to a non-square matrix A by use of the pseudo-inverse $A^\dagger$, as shown in Ref. [6]. It must be noticed that, for both the condition number and the RGA, their most important values are those close to the crossover.

4 Design procedure

As stated previously, the purpose of the proposed procedure is to simplify the transfer function shaping activity. At the same time, this allows the synthesized controller K to reduce the plant sensitivity to uncertainties. This is done by finding a pair of scaling matrices that minimize the condition number of the system over all the possible scalings.
The procedure is described in detail for the case of a square 2×2 plant, but with the aid of the succeeding remarks it can be easily extended to square plants of higher order or even to non-square plants.

1. The first step consists in the selection of a design frequency ω_d, typically at the desired crossover, and in the evaluation of $\Lambda(G(\omega_d))$, that is the RGA of G at that frequency.
2. For a 2×2 matrix, the minimized condition number is given by[9]

$$\gamma^\star(G) = \|\Lambda\|_{i1} + \sqrt{\|\Lambda\|_{i1}^2 - 1} \tag{5}$$

   where $\|\Lambda\|_{i1} = \max_j(\sum_i |\lambda_{ij}|)$ is the induced 1-norm of the RGA-matrix of G. Therefore, the designer can easily determine the minimized condition number of $G(\omega_d)$.
3. Define

$$H \triangleq \begin{bmatrix} 0 & G^{-1} \\ G & 0 \end{bmatrix} \tag{6}$$

   Since it has been proven[10] that

$$\sqrt{\gamma^\star(G)} = \min_{D_I, D_O} \overline{\sigma}(DHD^{-1}) \tag{7}$$

   with $D = \text{diag}\{D_I^{-1}, D_O\}$, the matter reduces to a minimization problem. In fact, by comparing 5 and 7 one concludes that, once evaluated $\gamma^\star$ with Eq. 5, it is sufficient to determine the elements of the scaling matrices D_I and D_O that simultaneously satisfy

$$\left(\sqrt{\gamma^\star(G(\omega_d))} - \overline{\sigma}(DH(\omega_d)D^{-1})\right)^2 \le \epsilon^2 \tag{8}$$

   with ϵ as small as required. The minimization problem stated in 8 can be solved using standard software. If necessary, one can impose some constraints on the function to be minimized, for example to ensure that the actuators will not exceed their actual capabilities. This causes the minimized or optimal condition number not to be recovered by the resulting scaling matrices, however a sub-optimal solution compatible with actuators requirements is found. The Matlab `fmincon` algorithm allows one to find the minimum of a constrained nonlinear multi-variable function.
4. The scaled plant $G^\star$ is now determined by

$$G^\star = D_O\, G\, D_I \tag{9}$$

   Because the optimization has been performed, by choice, at the desired crossover frequency, the singular values of $G^\star$ are guaranteed to be as aligned as possible in the bandwidth region. This result is achieved before the shaping activity and no further align matrix is then required (unless the selected weights W_1 and W_2 contain terms that heavily modify the slopes of the singular values, see below). Because $G^\star$ will be as well conditioned as possible, its robustness will be improved. This is particularly useful if the original G is ill-conditioned and presents control problems (it is however required that G has a sufficiently small minimized condition number).
5. It is now useful to reorder the rows and the columns in $G^\star$, thus rearranging inputs and outputs, to make the plant as diagonally dominant as possible. To do this, as suggested in Ref. [6], the RGA-number is useful. This procedure makes the design of the weighting functions easier, because it often allows one to choose diagonal pre- and post-compensators. If the smallest achievable RGA-number is too large, non-diagonal weights can however be determined through existing procedures, as that discussed in Ref. [11].

This approach, rigorous for 2×2 matrices, can be extended to square plants of order $m\times m$ by simply substituting Eq. 5 with

$$\|\Lambda(G)\|_m - \frac{1}{\gamma^\star(G)} \le \gamma^\star(G) \le \|\Lambda(G)\|_{\text{sum}} + k(m) \tag{10}$$

where $k(m)$ is a constant ($k(2) = 0$, $\|\Lambda\|_m = 2\max\{\|\Lambda\|_{i1}, \|\Lambda\|_{i\infty}\}$, where $\|\Lambda\|_{i\infty} = \max_i(\sum_j |\lambda_{ij}|)$ is the induced ∞-norm). The lower bound in 10 has been proved in Ref. [12], while the second one has been conjectured for $m > 2$:[12, 13] for example $k(3) = 1$ and $k(4) = 2$. The concept of pseudo-inverse matrix, useful to determine the RGA and to evaluate H in Eq. 6, allows one to extend the procedure to non-square plants.

The designer can now continue with the modeling of the singular values, by selecting the elements of the pre-compensator W_p that determine the dynamic shaping. W_p will contain integral action to obtain high gains at low frequencies, together with phase-advance to reduce the roll-off rate to 20dB/decade in the crossover region and phase-lag, if required, to increase the roll-off rate at high frequencies. After that, a left-coprime factor stabilization on G_S is performed, K_∞ is determined and the closed-loop is created.

Two important advantages arise from the procedure. First, it can be completely automatized by using existing software: the only work to be done by the designer is to choose an adequate crossover frequency at step 1 and, if required after some trials, to tune the constraints to the function to be minimized at step 3. Second, no align matrix W_a is required, being the alignment of singular values at the bandwidth already performed by the scaling matrices. Nor W_g is necessary, because the control against an excessive actuator usage is provided by the scaling matrices (recall that the constraints in the minimization process at step 3 are chosen in such a way to take into account the actuators capabilities). Moreover, unless it is required to prioritize some controlled variables over others (this seldom happens), W_2 can be set as the identity matrix. Therefore, all the constant terms that should be contained in the weighting matrices, result to be already included in the scaling matrices. Such matrices can be considered as parts of the pre- and post-compensators, or

$$W_1 = D_I W_p \quad (11)$$

$$W_2 = D_O \quad (12)$$

$$G_S = D_O G D_I W_p \quad (13)$$

Note that, as mentioned at step 4 of the procedure, it may happens that the dynamic shaping of the singular values lightly shifts their crossover frequency. In this case, the procedure can be repeated after having selected a more appropriate design frequency, otherwise singular values can be simply moved by slightly modifying the terms of W_2 or by multiplying W_1 by an adequate constant.

5 Case study

As a case study, we make use of a particular aircraft simulation model, called ADMIRE (Aerodata Model in Research Environment), and developed by the Swedish Defence Research Agency. ADMIRE is a non-linear, six degree of freedom simulation model of a rigid small fighter aircraft with a delta-canard configuration. Available control effectors are canards, leading edge flaps, elevons, rudder and throttle setting. The software package, entirely realized in Matlab/Simulink environment, contains routines suitable to trim the non-linear model inside its flight envelope and to obtain the corresponding linearized model. In this example we consider a trim condition characterized by a Mach number $M = 0.35$ and a height $h = 3000m$. The plant is well conditioned.

Our goal is to control both the flight velocity V and the pitch attitude angle θ by use of canards and elevons as inputs. The project requirements are given in terms of rise time: for both the outputs it is required to reach 90% of the commanded value in less than two seconds. At the same time it is desirable to obtain a good decoupling between the two channels. The model is expressed in its state-space form and in the control system design only the longitudinal states are considered $(V, \alpha, q, \theta, h)$ along with the states of the actuators (δ_c, δ_e).

The first task is the selection of an adequate bandwidth, enough to meet the desired performance, but limited by the speed of response of the actuators. The rise time requirement translates into a desired bandwidth of $4rad/s$ (to be used as ω_d) for both the loops.

A Matlab routine implementing the proposed design procedure evaluates the minimized condition number at $4rad/s$ (the result is $\gamma^\star(\omega_d) = 1.39$) and computes the input and output scaling matrices that approximate it with a tolerance $\epsilon = 10^{-6}$.

The minimization process at step 3 of the procedure is obtained by taking into account suitable constraints for both the inputs and the outputs. For the inputs, a maximum allowed deflection of $25deg$ for both canards and elevons has been imposed, while one unit of maximum admissible cross-coupling has been imposed on the outputs.

Figure 2 shows the original condition number of plant G (dotted line), the minimized condition number (dashed line) and the scaled system $G^\star$ condition number (solid line) as a function of frequency. Figure 3 shows the singular values

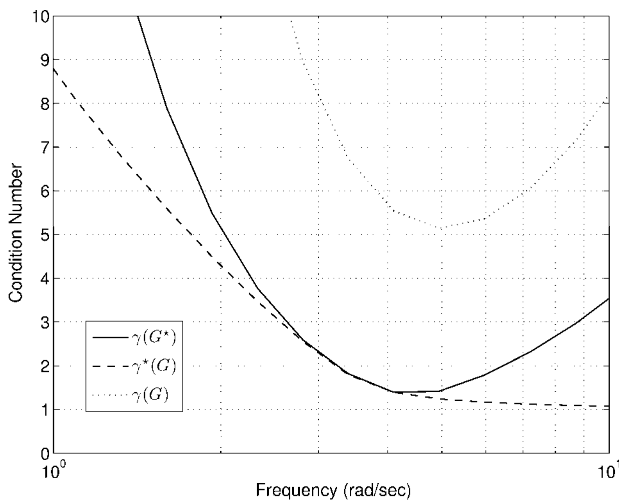

Fig. 2: Comparison between $\gamma(G)$, $\gamma^\star(G)$ and $\gamma(G^\star)$

of both G (dashed line) and $G^\star$ (solid line), before the dynamic weighting procedure. Note that the singular values of $G^\star$ are aligned in the crossover region.

The matrix W_p is chosen to obtain high gains at low frequencies, along with a reduction of roll-off rates in the crossover region and an increase of roll-off rates for higher

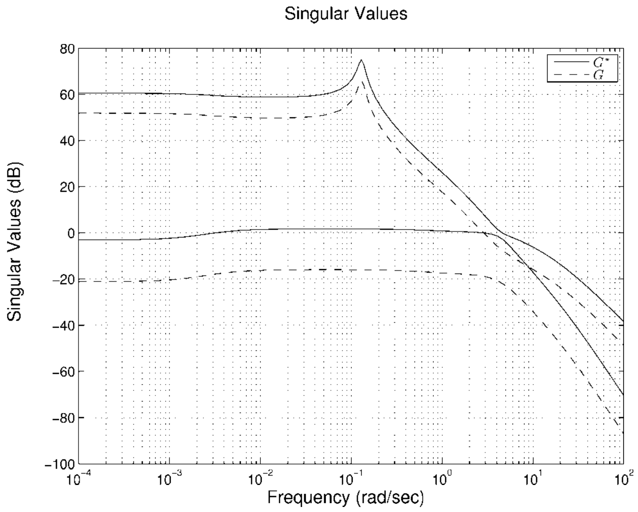

Fig. 3: Singular values of G and $G^{\star}$

frequencies. The choice is

$$W_p = \begin{bmatrix} \frac{s+0.5}{s} & 0 \\ 0 & \frac{s+1}{s} \end{bmatrix} \tag{14}$$

Figure 4 shows a comparison between the singular values of $G^{\star}$ and that of G_S, obtained after the dynamic shaping. Recall that W_a and W_g are not required and W_2 is the identity matrix.

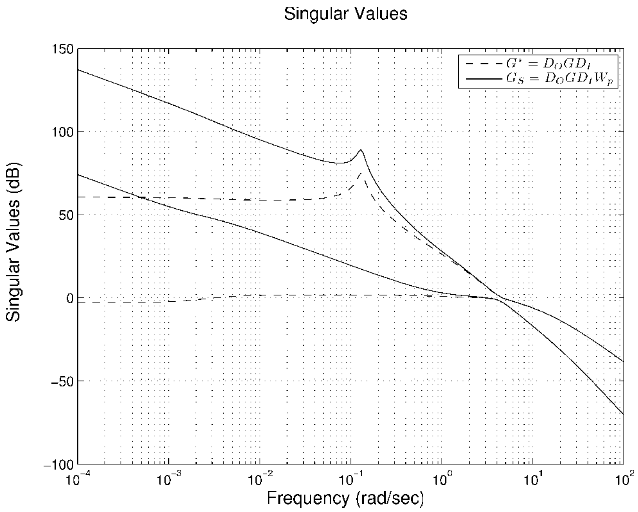

Fig. 4: Singular values of $G^{\star}$ and G_S

A left-coprime factor stabilization on G_S is performed and K_∞ is found using the Matlab function `ncfsyn`. The resulting K_∞ is compatible with the singular values of the shaped plant with a stability margin of 33%.
The closed-loop plant is verified by analyzing its response to step commands. Figures 5 and 6 show the step responses for commands applied respectively to $V^{\star}$ and to $\theta^{\star}$. The requirements in terms of rise time are satisfied in both cases and an adequate decoupling between the two channels is obtained.

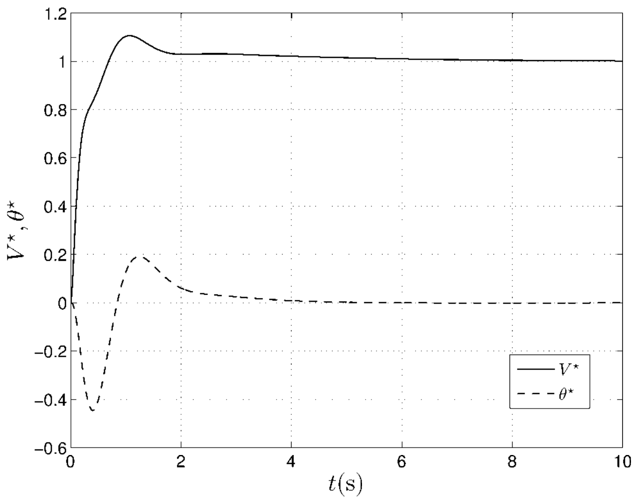

Fig. 5: Step response - command applied to $V^{\star}$

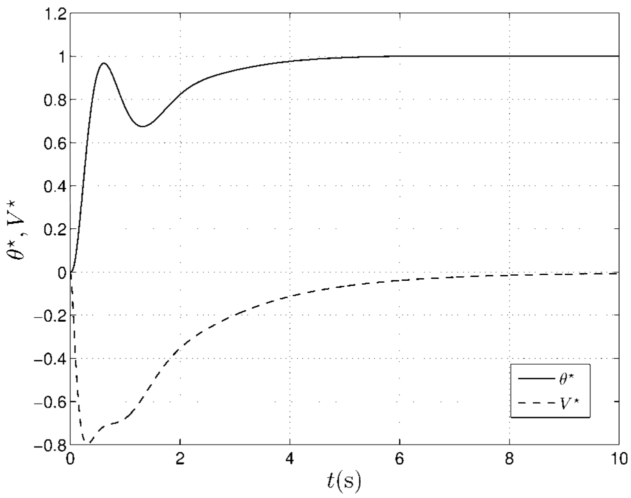

Fig. 6: Step response - command applied to $\theta^{\star}$

The actuator deflections corresponding to step commands of $V^{\star}$ and $\theta^{\star}$ are shown in Figures 7 and 8.

6 Conclusions

The proposed procedure simplifies the controller synthesis procedure by realizing an alignment of the singular values of the plant at the desired bandwidth before the choice of the weighting matrices. All the constant terms in such matrices are determined before the shaping and no further alignment is then required. This is particularly useful for plants characterized by a large original condition number or that experience ill-conditioning after traditional scaling methods (provided that they have a small minimized condition number). In fact, in this latter case the alignment of singular values at the crossover can be very difficult, especially after the shaping activity. Moreover, a condition number close to the minimized one can be recovered, especially in the

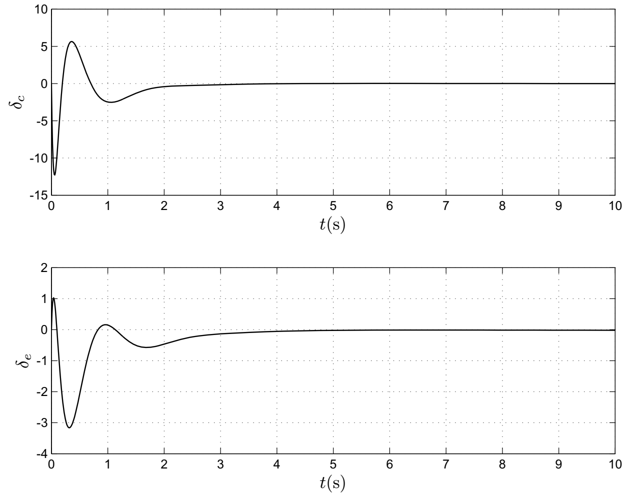

Fig. 7: Actuator deflections corresponding to a step command applied to $V^{\star}$

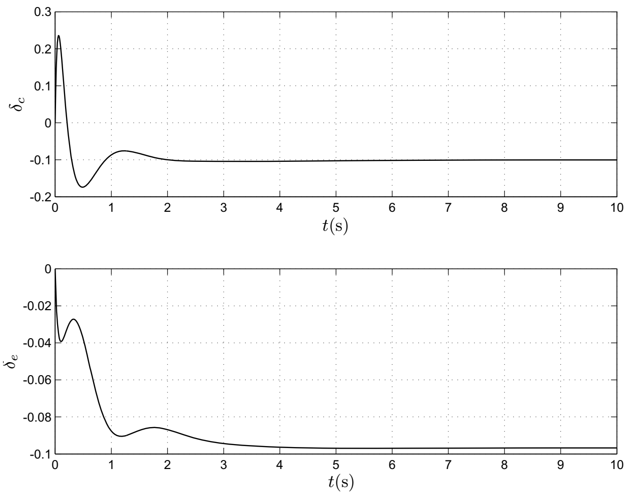

Fig. 8: Actuator deflections corresponding to a step command applied to $\theta^{\star}$

crossover region. This guarantees the plant to have a reduced sensitivity to model uncertainties.

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