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SYMMETRY ANALYSIS IN HYDRODYNAMIC STABILITY THEORY

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Abstract

For the last decades, hydrodynamic stability theory has been vastly studied using the normal-mode approach leading to the well-known Orr-Sommerfeld Equation. Though rather successful for boundary layer flows it essentially failed for internal canonical flows such as the channel, Couette or pipe flow. In order to generate new approaches which are inherently given in the mathematical formulation of the problem, we apply symmetry methods on the linearized Navier-Stokes Equation (LNSE) for nearly parallel shear flows with two-dimensional perturbations. In particular, we show that the Orr-Sommerfeld-Equation is a symmetry reduction using the full set of symmetries for unrestricted base flows. We found that restricting the base flow to a special profile, the LNSE admits further symmetries, which we present in the context of a complete symmetry classification. Especially the algebraic, the exponential, the logarithmic and the linear shear flow admit additional symmetries, for which we present new solutions of the perturbations which are very distinct from the normal mode solution. In particular, we show that the generally known solution for the stability problem of linear viscous base flows is based on a symmetry reduction, too. We give an alternative formulation to this solution, using a different set of symmetries. For an inviscid linear shear flow, we give a simple formulation of a solution consisting of stable self-similar modes.

1. INTRODUCTION

The nature of the transition from a laminar flow to turbulence is one of the greatest unresolved mysteries of fluid mechanics. It is of paramount interest for many industrial fields of applications, e.g. for the aviation industry and for natural phenomena as in meteorology. Consequently, this topic has received considerable attention during the last decades.

In particular, the stability of nearly parallel shear flows, which is of interest for jets, wakes and shear layers, has been investigated abundantly. Most commonly, a normal mode approach has been applied on the linearized Navier-Stokes-Equations (LNSE) for the two-dimensional perturbation of a laminar base flow, leading to the famous Orr-Sommerfeld-Equation (OSE). Though rather successful in boundary layer flows, the OSE does not give proper results for some particular base flows, especially for the plane channel flow with the classical parabolic flow profile.

In this work, we apply symmetry methods on the LNSE in order to systematically derive new approaches for solving the stability problem for nearly parallel shear flows. In the field of fluid mechanics experience has shown that many flows turn out to be invariant under certain symmetries. In the present case, we first show that the OSE is solely based on three symmetries of the LNSE. In fact, the OSE is a symmetry reduction using the latter three symmetries. Secondly, we do a complete symmetry-classification for the LNSE in order to find further self-similar ansatz functions for special base flows, which are then used to simplify the LNSE.

In the inviscid case, we found additional symmetries for parabolic, exponential, logarithmic and linear base flows. In all cases, the symmetries lead to new ansatz functions considerably distinct from the normal mode ansatz leading to the OSE. We investigate the parabolic base flow in more detail and analyze techniques to solve the equation and to fulfill the boundary conditions. For the plane Couette flow, we show that the generally known solution is based on the three basic symmetries of the LNSE. Using the extended set of symmetries, we present an alternative formulation of a new general solution. In the inviscid case, this leads to a closed formulation of stable self-similar modes.

This work is organized as follows: In section 2 and 3, we introduce the basic equations and present a brief introduction to symmetry methods. A systematic derivation of the Orr-Sommerfeld-equation using symmetry methods is given in section 4, whereas the complete symmetry classification of the equation in case is given in section 5. Special base flow profiles together with their additional symmetries and the respective similarity solutions are presented in sections 6, 7 and 8, where the parabolic channel flow, non-algebraic base flows and the linear shear flow are discussed, respectively.

2. PROBLEM FORMULATION

We consider a parallel base flow $(U(y),0,0)^T$ with a two-dimensional perturbation of the form $(u(x,y,t), v(x,y,t),0)^T$. It is assumed that the Navier-Stokes-Equation holds for the base flow, which is subtracted from the Navier-Stokes-Equation for the perturbed flow $(U(y)+u(x,y,t),v(x,y,t),0)^T$. As we are dealing with two-dimensional flows, one can introduce a stream function $\psi(x,y,t)$ such that

$$(1) \quad u=\frac{\partial\psi}{\partial y} \quad \text{and} \quad v=-\frac{\partial\psi}{\partial x},$$

which leads to the elimination of the continuity equation. Linearizing the resulting equation leads to the non-dimensionalized partial differential equation of fourth order

(2) $\frac{\partial}{\partial t}\Delta\psi - \frac{dU}{dy^2}\frac{\partial\psi}{\partial x} + U(y)\Delta\psi = \frac{1}{\mathrm{Re}}\Delta\Delta\psi,$

where Re is the Reynolds number of the system.

3. SYMMETRY ANALYSIS

A comprehensive review of symmetry analysis is given by Bluman and Kumei [1] and by Steeb [2]. For clarification, we give a brief introduction to the main ideas of this topic, which is mainly used in order to solve differential equations of the form

(3) $\boldsymbol{F}(\boldsymbol{x},\boldsymbol{u},\boldsymbol{u}_{[1]},\boldsymbol{u}_{[2]},\dots) = \boldsymbol{0},$

where $\boldsymbol{x}$ is a vector of independent variables, $\boldsymbol{u}$ is a vector of dependent variables and $\boldsymbol{u}_{[n]}$ is the collection of the first n derivatives of $\boldsymbol{u}$ with respect to $\boldsymbol{x}$. In our case, $\boldsymbol{x} = (\boldsymbol{x},\boldsymbol{y},\boldsymbol{t})$, $\boldsymbol{u} = (\boldsymbol{\psi}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{t}))$ and $\boldsymbol{F}$ is given by (2). A Lie-point-Symmetry is a transformation of the type

(4) $\tilde{\boldsymbol{x}} = \boldsymbol{\Phi}(\boldsymbol{x},\boldsymbol{u},\boldsymbol{\varepsilon}) = \boldsymbol{x} + \epsilon\boldsymbol{\xi}(\boldsymbol{x},\boldsymbol{u}) + \boldsymbol{O}(\epsilon^2)$

(5) $\tilde{\boldsymbol{u}} = \boldsymbol{\Psi}(\boldsymbol{x},\boldsymbol{u},\boldsymbol{\varepsilon}) = \boldsymbol{u} + \epsilon\boldsymbol{\eta}(\boldsymbol{x},\boldsymbol{u}) + \boldsymbol{O}(\epsilon^2)$

with

$$\boldsymbol{\xi}(\boldsymbol{x},\boldsymbol{u}) := \left.\frac{\partial\boldsymbol{\Phi}(\boldsymbol{x},\boldsymbol{u},\boldsymbol{\varepsilon})}{\partial\boldsymbol{\epsilon}}\right|_{\boldsymbol{\epsilon}=0}, \quad \text{and}$$

$$\boldsymbol{\eta}(\boldsymbol{x},\boldsymbol{u}) := \left.\frac{\partial\boldsymbol{\Psi}(\boldsymbol{x},\boldsymbol{u},\boldsymbol{\varepsilon})}{\partial\boldsymbol{\epsilon}}\right|_{\boldsymbol{\epsilon}=0},$$

such that $\boldsymbol{u}(\boldsymbol{x})$ as well as $\tilde{\boldsymbol{u}}(\tilde{\boldsymbol{x}})$ are both solutions of the differential equation [3]. A solution is said to be invariant under a Lie-Symmetry transformation, if $\tilde{\boldsymbol{u}}(\tilde{\boldsymbol{x}}) = \boldsymbol{u}(\boldsymbol{x})$. These solutions are also denoted as *similarity solutions*. Lie's fundamental theorem states that Lie transformations are uniquely defined by their (local) infinitesimals $\boldsymbol{\xi}(\boldsymbol{x},\boldsymbol{u})$ and $\boldsymbol{\eta}(\boldsymbol{x},\boldsymbol{u})$ [3]. They can be identified as the components of the following scalar tangent field in $(\boldsymbol{x},\boldsymbol{u})$-space, also known as the infinitesimal generator:

(6) $X := \xi_i(\boldsymbol{x},\boldsymbol{u})\frac{\partial}{\partial x_i} + \eta_i(\boldsymbol{x},\boldsymbol{u})\frac{\partial}{\partial u_i}.$

The basic assumption which enables us to do so is that the transformation (4)-(5) has a group structure. In this work, symmetries will mostly be identified by their infinitesimal generator (6). The symmetries of the differential equations are calculated by means of the Lie-Algorithmc [3], where we have used the GEM package of Cheviakov [4] and the DESOLVE package of Vu and Carminati [5].

4. THE ORIGIN OF THE ORR-SOMMERFELD-EQUATION

For general base flows $U(y)$, equation (2) admits four symmetries: Superposition (X_0), translation in x-direction (X_1) and in time (X_2) and scaling of ψ (X_3), as shown in TAB 1.

Infinitesimal Generator		Transformation
$X_0 = f(x,y,t)\frac{\partial}{\partial\psi}$	$\Leftrightarrow$	$[\tilde{x} = x, \tilde{y} = y, \tilde{t} = t, \tilde{\psi} = \psi + f(x,y,t)]$
$X_1 = \frac{\partial}{\partial x}$	$\Leftrightarrow$	$[\tilde{x} = x + x_0, \tilde{y} = y, \tilde{t} = t, \tilde{\psi} = \psi]$
$X_2 = \frac{\partial}{\partial t}$	$\Leftrightarrow$	$[\tilde{x} = x, \tilde{y} = y, \tilde{t} = t + t_0, \tilde{\psi} = \psi]$
$X_3 = \psi\frac{\partial}{\partial\psi}$	$\Leftrightarrow$	$[\tilde{x} = x, \tilde{y} = y, \tilde{t} = t, \tilde{\psi} = C\psi]$

TAB 1 : Infinitesimal Generators of the symmetries of equation (2) for an unrestricted base flow $U(y)$, where $\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{t})$ is a solution of the equation under investigation, here (2) and $\boldsymbol{x}_0$, $\boldsymbol{t}_0$ and $\boldsymbol{C}$ are arbitrary scalar values..

Due to linearity of (2), it is allowed to superpose the transformations (see also X_0). Hence, the general symmetry can be formulated as a combination of the remaining symmetries X_1-X_3:

(7) $X = \alpha X_1 + X_2 + \gamma X_3$

with $\alpha,\gamma \in \mathbb{C}$. The respective invariant solution is given by

(8) $\psi(x,y,t) = f(x - \alpha t, y)e^{\gamma t}.$

Inserting (8) into (2) gives a necessary condition for the function $f(\zeta,y)$:

(9) $(U(y) - \alpha)\frac{\partial}{\partial\zeta}\Delta f + \gamma\Delta f - U''(y)\frac{\partial}{\partial\zeta}f = \frac{1}{\mathrm{Re}}\Delta\Delta f.$

This equation admits the scaling symmetry $\tilde{X}_1 = f\frac{\partial}{\partial f}$ as well as the translation symmetry $\tilde{X}_2 = \frac{\partial}{\partial\zeta}$. We repeat the procedure leading to (8) by choosing f to be invariant under the general symmetry

$$\delta\tilde{X}_1 + \tilde{X}_2$$

of equation (9), where $\delta \in \mathbb{C}$. The invariant solution to this symmetry is

(10) $f(\zeta,y) = \tilde{f}(y)e^{\delta\zeta}.$

Inserting this approach into (8) and renaming the coefficients by

$$\tilde{\alpha} := -i\delta$$

$$c := \frac{\alpha - \gamma}{\delta}$$

yields the following normal-mode approach:

$$\psi(x,y,t) = \tilde{f}(y)e^{i\tilde{\alpha}(x-ct)},$$

which by insertion into (2) finally leads to the famous Orr-Sommerfeld-Equation:

$$(U(y) - c)\left(\frac{d^2}{dy^2} - \tilde{\alpha}^2\right)\tilde{f} = \frac{1}{i\mathrm{Re}\tilde{\alpha}}\left(\frac{d^2}{dy^2} - \tilde{\alpha}^2\right)^2\tilde{f}.$$

In other words, the OSE is derived from a successive reduction of the linearised Navier-Stokes-Equation each time using the full set of admissible symmetries. Note that this holds true for both the viscous and the inviscid case and is usually referred to as the normal mode or modal Ansatz.

5. COMPLETE SYMMETRY CLASSIFICATION

In the previous section, we have presented the symmetries of equation (2) for an unrestricted base flow $U(y)$. Restricting $U(y)$ to special base flows amplifies the number of symmetries of (2). Here, we perform a complete symmetry classification, i.e. we derive all base flows

$U(y)$ which allow additional symmetries. In TAB 2, all additional base flows are given together with the infinitesimal generators of the symmetries they admit in the inviscid case ($\mathrm{Re}=\infty$). In the viscous case, equation (2) only allows two special base flows with additional symmetries. Particularly, the linear shear flow $U(y)=ay+b$ only allows one symmetry. Furthermore, the algebraic base flow $U(y)=a(y+b)^c+d$ which allows one additional symmetry in the inviscid case reduces to the case $c=-1$ in the viscous case (see TAB 2 and TAB 3).

$U(y)$	Additional Symmetries
$ay+b$	$X_4^{lin}=at\frac{\partial}{\partial x}+\frac{\partial}{\partial y}$ $X_5^{lin}=(x-tb)\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}$
$a(y+b)^c+d$	$X_4^{alg}=(cdt-x)\frac{\partial}{\partial x}-(y+b)\frac{\partial}{\partial y}+(c-1)t\frac{\partial}{\partial t}$
$ae^{by}+c$	$X_4^{exp}=-bct\frac{\partial}{\partial x}+\frac{\partial}{\partial y}-bt\frac{\partial}{\partial t}$
$a\ln(y+b)+c$	$X_4^{log}=(x+at)\frac{\partial}{\partial x}+(y+b)\frac{\partial}{\partial y}+t\frac{\partial}{\partial t}$

TAB 2 : Infinitesimal Generators of the symmetries of equation (2) for special base flows $U(y)$ in the inviscid case ($\mathrm{Re}=\infty$). Additional to the symmetries given here, all base flows allow the basic symmetries X_0-X_3 given in TAB 1. a,b,c and d are arbitrary real numbers.

$U(y)$	Additional Symmetries
$ay+b$	$X_4^{lin}=at\frac{\partial}{\partial x}+\frac{\partial}{\partial y}$
$\frac{a}{y+b}+d$	$X_4^{alg,vis}=-(ct+x)\frac{\partial}{\partial x}-(y+b)\frac{\partial}{\partial y}-2t\frac{\partial}{\partial t}$

TAB 3 Infinitesimal Generators of the symmetries of equation (2) for special base flows $U(y)$ in the viscous case ($\mathrm{Re}<\infty$). Additional to the symmetries given here, all base flows allow the basic symmetries X_0-X_3 given in TAB 1. a,b and d are arbitrary real numbers.

6. THE PARABOLIC BASE FLOW

The parabolic channel flow of the form

$$U:[-1,1]\to\mathbb{R}\,,\qquad y\to(1-y^2)$$

is a special case of the algebraic base flow in TAB 2 ($c=2$). The infinitesimal generator of the general symmetry to this base flow is a linear combination of the three basic symmetries X_1-X_3 plus the additional symmetry X_4^{alg} (see TAB 1 and TAB 2):

$$(11)\; X := \alpha X_1+\delta X_2+\beta X_3+X_4^{alg}$$

It can be shown that all solutions which are invariant with respect to (11) can be written as

$$(12)\; \psi(x,y,t) = (t+\delta)^{\beta} f(\zeta,\theta)$$

with

$$(13)\; \zeta := (t+\delta)(x-t+\delta-\alpha) \text{ and}$$

$$(14)\; \theta := y(t+\delta).$$

Inserting this approach into equation (2) gives the following necessary condition for the function $f(\zeta,\theta)$:

$$\left((\zeta-\theta^2)\frac{\partial}{\partial\zeta}+\theta\frac{\partial}{\partial\theta}+(2+\beta)\right)\Delta f+2\frac{\partial f}{\partial\xi}=0.$$

This partial differential equation of third order has no further symmetries except the scaling symmetry in $f(\zeta,\theta)$. Remark that because of (13) and (14) $f(\zeta,\theta)$ is constant on the paths given by

$$(15)\; \begin{pmatrix}x(t)\\y(t)\end{pmatrix}=\begin{pmatrix}t+\frac{\zeta_0}{t+\delta}-\delta+\alpha\\ \frac{\theta_0}{t+\delta}\end{pmatrix}.$$

Here, the linear component in $x(t)$ suggests a travelling-wave solution. Furthermore, the component $\sim(t+\delta)^{-1}$ suggests a solution with steepening gradients, depending on the parameter β. Inserting (12) into (1) shows that on the path given by (15), the amplitude of the velocities u and v increases with $\sim(t+\delta)^{\beta+1}$. Consequently, depending on the symmetry parameter β, an algebraic growth or decay of the velocities can be observed.

7. NON-ALGEBRAIC BASE FLOWS

7.1. The logarithmic base flow

The infinitesimal generator of the general symmetry for a logarithmic base flow $U(y)=a\ln(y+b)+c$ in TAB 2 can be written as the linear combination

$$(16)\; \alpha X_1+\delta X_2+\beta X_3+X_4^{\log}.$$

Invariant solutions of equation (2) with respect to this symmetry have the following form:

$$\psi(x,y,t) = (t+\delta)^{\beta} f(\zeta,\theta)$$

with

$$\zeta := \frac{x+at+\alpha}{t+\delta}-a\ln(t+\delta)-c \quad\text{and}$$

$$\theta := \frac{y+b}{t+\delta}.$$

Inserting this approach into equation (2) gives a necessary condition for the function $f(\zeta,\theta)$:

$$(17)\; \theta^2\left((a\ln(\theta)-\zeta)\frac{\partial}{\partial\zeta}-\theta\frac{\partial}{\partial\theta}+(\beta-2)\right)\Delta f+ a\frac{\partial f}{\partial\xi}=0.$$

In this approach, the value of $f(\zeta,\theta)$ is constant on the paths given by

$$\begin{pmatrix}x(t)\\y(t)\end{pmatrix}=\begin{pmatrix}(x_0+\alpha)\left(1+\frac{t}{\delta}\right)+a\delta\ln\left(1+\frac{t}{\delta}\right)-at-\alpha\\ (y_0+b)\left(1+\frac{t}{\delta}\right)-b\end{pmatrix},$$

where we chose x_0 and y_0 such that the variables ζ and θ evaluated at (x_0,y_0) for $t=0$ equal the values of ζ and θ evaluated at $(x(t),y(t))$ for $t>0$, respectively. By the path given above, we get that the symmetry (16) suggests that a perturbation of the form $u(x_0,0) = A\sin(k_x x_0)$ propagates as a travelling wave with wave number $k_x{}^*(t)$ and phase offset $\alpha_x{}^*(t)$:

$$k_x{}^*(t) := k_x\left(1+\frac{t}{\delta}\right)^{-1}$$

$$\alpha_x{}^*(t) := k_x\left(\frac{at+\alpha}{1+\frac{t}{\delta}} - a\delta\ln\left(1+\frac{t}{\delta}\right) - \alpha\right)$$

We conclude that for periodical perturbations on a logarithmic base flow, symmetry methods suggest a linear increase of the wavelength $(k_x^*)^{-1}$ with time.

7.2. The exponential base flow

The exponential base flow in TAB 2 is restricted to a flow of the type

$$U(y) = c - ae^{-by},$$

where $a, b, c > 0$. Assuming that the wall is at $y = 0$, this corresponds with an exponential boundary layer type of flow. The infinitesimal generator of the general symmetry is, analogously to (16):

$$(18)\ \alpha X_1 + \delta X_2 + \beta X_3 + X_4^{\exp}.$$

Invariant solutions of equation (2) with respect to this symmetry have the form:

$$\psi(x,y,t) = e^{\beta y} f(\zeta,\theta)$$

with

$$\zeta := x - ct + \ln(\delta+bt)^{\frac{c\delta-\alpha}{b}} \quad \text{and}$$

$$\theta := y - \frac{\ln(\delta+bt)}{b},$$

where $\delta > 0$. As in the previous section, the necessary condition for the function $f(\zeta,\theta)$ is obtained by inserting this approach into equation (2):

$$\left(\left(\alpha - c\delta + ae^{-b\theta}\right)\frac{\partial}{\partial\zeta} + \frac{\partial}{\partial\theta}\right)\left(\frac{\partial^2}{\partial\zeta^2} + \left(\frac{\partial}{\partial\theta}+\beta\right)^2\right)f$$
$$- -ab^2e^{b\theta}\frac{\partial}{\partial\zeta}f = 0.$$

Here, the path $(x(t), y(t))$ on which $f(\zeta,\theta)$ is constant suggests that a perturbation moves logarithmically in positive y -direction, away from the wall:

$$\begin{pmatrix}x(t)\\y(t)\end{pmatrix} = \begin{pmatrix}x_0 - \frac{c\delta-\alpha}{b}\ln\left(1+\frac{b}{\delta}t\right) + ct\\ y_0 + \frac{1}{b}\ln\left(1+\frac{b}{\delta}t\right)\end{pmatrix}.$$

8. LINEAR SHEAR FLOW

We consider a linear shear flow profile of the form $U(y) = ay$. In the viscous case, this equation admits four Lie-Point-Symmetries. Additionally to the three basic symmetries $X_1 - X_3$ (see also TAB 1), it admits the symmetry $X_4^{lin} = at\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$, for which the equivalent variable transformation can be written as

$$[\tilde{x} = x + Cat, \tilde{y} = y + C, \tilde{t} = t, \tilde{\psi} = \psi].$$

The general infinitesimal generator for the viscous case can be written as:

$$(19)\ \alpha X_1 + \delta X_2 + \beta X_3 + X_4^{\text{lin}}.$$

In the inviscid case, there is the additional symmetry $X_5^{lin} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ with the variable transformation

$$[\tilde{x} = C(x-tb) + tb, \tilde{y} = Cy, \tilde{t} = t, \tilde{\psi} = \psi],$$

such that the infinitesimal generator of the general symmetry yields

$$(20)\ \alpha X_1 + \delta X_2 + \beta X_3 + X_4^{\text{lin}} + \gamma X_5^{\text{lin}}.$$

8.1. The viscous case

First, we search for a solution of equation (2) which is invariant with respect to the linear combination (19) where δ is set to zero. All similarity solutions to this symmetry can be written in the form

$$(21)\ \psi(x,y,t) = e^{\beta y} f(\zeta,\theta)$$

with

$$\zeta := x - y(\alpha + at) \quad \text{and}$$

$$\theta := \alpha + at.$$

The necessary condition for $f(\zeta,\theta)$ is given by

$$(22)\ D_1 D_2 f(\zeta,\theta) = \frac{1}{Re} D_2 D_2 f(\zeta,\theta),$$

where the differential operators D_1 and D_2 are defined as follows:

$$D_1 := a\frac{\partial}{\partial\theta}$$

$$D_2 := \left(\left(\beta - \theta\frac{\partial}{\partial\zeta}\right)^2 + \frac{\partial^2}{\partial\zeta^2}\right).$$

Similar to the derivation of the Orr-Sommerfeld-Equation, we perform a symmetry analysis of equation (22), obtaining the two symmetries $\tilde{X}_1 = f\frac{\partial}{\partial f}$ and $\tilde{X}_2 = \frac{\partial}{\partial\zeta}$. The invariant solution with respect to the linear combination

$$(23)\ \tilde{\beta}\tilde{X}_1 + \tilde{X}_2$$

can be written as

$$(24)\ f(\zeta,\theta) = \tilde{f}(\theta)e^{\tilde{\beta}\zeta}.$$

Inserting this approach into (21) gives

$$(25)\ \psi(x,y,t) = g(t)\exp(\kappa y + \lambda(x - ayt)),$$

where $= \tilde{\beta}$, $\kappa = \beta - \tilde{\beta}\alpha$ and $g(t)$ solves the differential equation

$$\left(b\lambda + \frac{d}{dt}\right)(\lambda^2 + (\lambda at - \kappa)^2)g(t) =$$
$$= \frac{1}{\text{Re}}(\lambda^2 + (\lambda at - \kappa)^2)^2 g(t).$$

One can easily show that the general solution of this ordinary differential equation of first order is

$$(26)\ g(t) = \frac{1}{\lambda^2 + (\lambda at - \kappa)^2}\exp\left(\frac{t}{\text{Re}}\left(\frac{1}{3}(\lambda at)^2 - \kappa\lambda at + \right.\right.$$
$$\left.\left. +\kappa^2 + \lambda^2\right) - b\lambda t\right).$$

Inserting (26) into (25) gives an invariant solution for the stream function ψ. Note that (2) is a linear equation. Consequently several invariant solutions can be superposed, which leads to the more general formulation

$$(27)\ \psi(x,y,t) = \iint \frac{c(\kappa,\lambda)}{\lambda^2+(\lambda a t-\kappa)^2} \times$$

$$\times\ e^{\frac{t}{\mathrm{Re}}\left(\frac{1}{3}(\lambda a t)^2-\kappa\lambda a t+\kappa^2+\lambda^2\right)-b\lambda t} e^{\kappa y+\lambda(x-ayt)}\, d\kappa d\lambda.$$

This solution corresponds with the generally known solution of equation (2) for the case of a linear viscous shear flow (see also Moffatt [6]).

A different approach is obtained if δ in (19) is unequal zero, i.e. if the full general combination of symmetries is used. This leads to the following invariant solution:

$$(28)\ \psi(x,y,t) = e^{\frac{\beta}{\delta}y} f(\zeta,\theta)$$

with

$$\zeta := x - \frac{\alpha}{\delta}t - \frac{a}{2\delta}t^2 \text{ and}$$

$$\theta := y - \frac{t}{\delta} - \frac{\alpha}{a\delta},$$

where $f(\zeta,\theta)$ solves the partial differential equation

$$(29)\ \left(a\delta\eta\frac{\partial}{\partial\zeta} - \frac{\partial}{\partial\theta} + \beta\right)\Delta f(\zeta,\theta) = \frac{\delta}{\mathrm{Re}}\Delta\Delta f(\zeta,\theta).$$

As in the previous sections, this equations is obtained by inserting (28) into (2). Equation (29) admits one scaling and one translational symmetry such that the infinitesimal generator of the linear combination of symmetries can be written as

$$\tilde{\beta}\tilde{X}_1 + \tilde{X}_2$$

with

$$\tilde{X}_1 = f\frac{\partial}{\partial f} \text{ and } \tilde{X}_2 = \frac{\partial}{\partial\zeta}.$$

The resulting invariant solution is analogous to (24). It can be inserted into (28) such that we obtain the approach

$$(30)\ \psi(x,y,t) = g(\vartheta)\exp\left(i\kappa\left(x - \frac{at^2}{2\delta}\right) + \lambda t\right)$$

with

$$\vartheta = y - \frac{t}{\delta},$$

and the parameters

$$\kappa = -i\tilde{\beta} \text{ and } \lambda = \left(\frac{\beta}{\delta} - \frac{\tilde{\beta}\alpha}{\delta}\right).$$

Remark that κ has to be a purely real number. If it had an imaginary part, then ψ would grow exponentially for $x \to \infty$, which leads to a nonphysical solution. Furthermore, we assume that the scaling factor δ for the time t is a real number. Inserting (30) into equation (2) gives the following partial differential equation for $g(\vartheta)$:

$$(31)\ \left(-R_\delta\frac{d}{d\tilde{\vartheta}} + (R_\lambda + i\tilde{\vartheta})\right)\left(\frac{d^2}{d\tilde{\vartheta}^2} - 1\right)g(\tilde{\vartheta}) = \frac{1}{\mathrm{Re}_a}\left(\frac{d^2}{d\tilde{\vartheta}^2} - 1\right)^2 g(\tilde{\vartheta})$$

with

$$\tilde{\vartheta} = \kappa\vartheta$$

and

$$\mathrm{Re}_a = \frac{a}{\kappa^2}\mathrm{Re}\ ,\ \mathrm{Re}_a \in \mathbb{R}^+,$$

$$\mathrm{R}_\delta = \frac{\kappa}{\delta a}\ ,\ \mathrm{R}_\delta \in \mathbb{R},$$

$$\mathrm{R}_\lambda = \frac{\lambda}{a}\ ,\ \mathrm{R}_\lambda \in \mathbb{C}.$$

One can show that the general solution of equation (31) can be written as a linear combination of the following basis functions [7]:

$$g_1(\tilde{\vartheta}) = e^{\tilde{\vartheta}},$$

$$g_2(\tilde{\vartheta}) = e^{-\tilde{\vartheta}},$$

$$g_3(\tilde{\vartheta}) = \int_0^{\tilde{\vartheta}} \sinh(\tilde{\vartheta} - s)\, e^{-\frac{\mathrm{Re}_a\mathrm{R}_\delta}{2}s}\mathrm{Ai}(a_1 s + a_2)ds$$

$$g_3(\tilde{\vartheta}) = \int_0^{\tilde{\vartheta}} \sinh(\tilde{\vartheta} - s)\, e^{-\frac{\mathrm{Re}_a\mathrm{R}_\delta}{2}s}\mathrm{Ai}\left((a_1 s + a_2)e^{2\pi i/3}\right)ds$$

where the parameters are given by

$$a_1 = e^{i\pi/6}\mathrm{Re}_a^{1/3}$$

$$\text{and } a_2 = e^{-\frac{i\pi}{3}}\left(1 + \mathrm{Re}_a\mathrm{R}_\lambda + \left(\frac{\mathrm{Re}_a\mathrm{R}_\delta}{2}\right)^2\right),$$

where Ai is the Airy function. The link between this novel formulation of the solution and the known solution (27) remains to be shown.

8.2. The inviscid case

We consider solutions of equation (2) in the inviscid case which are invariant with respect to the symmetry (20). It can be shown that all invariant solutions can be reduced to three qualitatively distinct cases, out of which we present one case which leads to a closed formulation of stable modes, where $\delta \neq 0$ and $\gamma = 0$. For the other cases, see also Nold [7].

Note that setting γ to zero leads to the symmetry as in (19), which leads to equation (31). In the inviscid case, this equation yields

$$(32)\ \left(-R_\delta\frac{d}{d\tilde{\vartheta}} + (R_\lambda + i\tilde{\vartheta})\right)\left(\frac{d^2}{d\tilde{\vartheta}^2} - 1\right)g(\tilde{\vartheta}) = 0.$$

The solution of this equation can be written in Fourier-space as

$$(33)\ \hat{g}(w) = -C\frac{\exp\left(R_\lambda w - i\frac{R_\delta}{2}w^2\right)}{1+w^2},$$

where $\hat{g}(w)$ is the Fourier-transform of $g(\tilde{\vartheta})$. (33) is now split into the three factors

$$\hat{g}_1(w) = \exp(iR_{\lambda,I}w)$$

$$\hat{g}_2(w) = \exp\left(-i\frac{R_\delta}{2}w^2\right)$$

$$\hat{g}_3(w) = \frac{-1}{1+w^2},$$

where we assumed that R_λ is a purely imaginary number $R_\lambda = i\,R_{\lambda,I}$, where $R_{\lambda,I} \in \mathbb{R}$. Note that otherwise, $\hat{g}_1(w)$ would not be well-defined. We subsequently transform the functions $\hat{g}_{1,2,3}(w)$ from Fourier space back into real space:

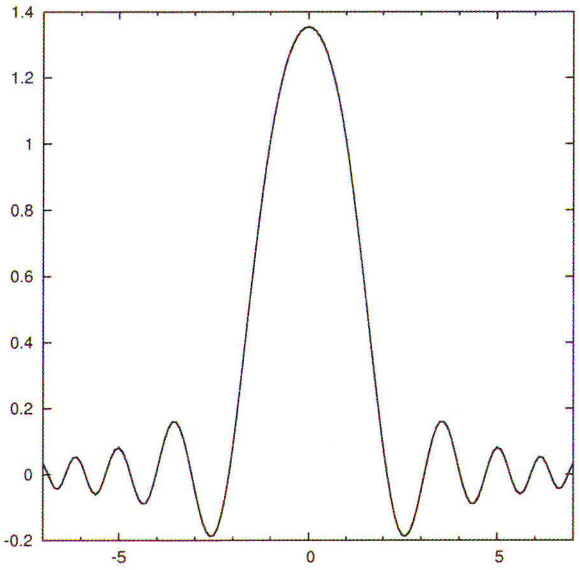

FIG 1 . Plot of the real part of $g(\tilde{\vartheta})$ in (34) over $\tilde{\vartheta}$ for parameters $R_\delta = 1$ and $R_{\lambda,I} = 0$. The graph shows a maximum at $\tilde{\vartheta} = 0$ and an oscillatory decay for $\tilde{\vartheta} \to \pm\infty$.

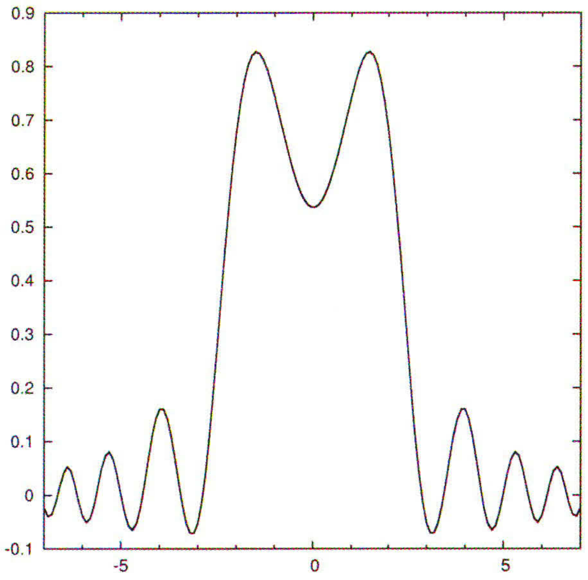

FIG 2 . Plot of the imaginary part of $g(\tilde{\vartheta})$ in (34) over $\tilde{\vartheta}$ for parameters $R_\delta = 1$ and $R_{\lambda,I} = 0$. As in FIG 1, the graph shows an oscillatory decay for $\tilde{\vartheta} \to \pm\infty$.

$$g_1(\tilde{\vartheta}) = \sqrt{2\pi}\,\delta(\tilde{\vartheta} - R_{\lambda,I}),$$

$$g_2(\tilde{\vartheta}) = \sqrt{\frac{1}{2R_\delta}}(1-i)\exp\left(\frac{i\tilde{\vartheta}^2}{2R_\delta}\right),$$

$$g_3(\tilde{\vartheta}) = -\sqrt{\frac{\pi}{2}}\exp(-|\tilde{\vartheta}|).$$

Taking the convolution

$$g_2(\tilde{\vartheta}) * g_1(\tilde{\vartheta}) = \exp\left(i\frac{(\tilde{\vartheta} - R_{\lambda,I})^2}{2R_\delta}\right)$$

finally leads to an expression for $g(\tilde{\vartheta})$ in the real space:

$$(34)\; g(\tilde{\vartheta}) = \left(g_2(\tilde{\vartheta}) * g_1(\tilde{\vartheta})\right) * g_3(\tilde{\vartheta})$$
$$= T_{R_{\lambda,I}}\left(\int_{-\infty}^{\infty} \exp\left(i\frac{\tau^2}{2R_\delta} - |\tilde{\vartheta} - \tau|\right) d\tau\right),$$

where the operator T_R defines a translation as:

$$T_R(f(\cdot)) := f(\cdot - R).$$

Note that by (34), the parameter $R_{\lambda,I}$ only induces a shift in $\tilde{\vartheta}$. In FIG 1 and FIG 2 we plot the real and the imaginary part of (34) for $R_{\lambda,I} = 0$ and for $R_\delta = 1$. Inserting (34) into (30) yields a closed expression for a self-similar solution of equation (2) in the inviscid case.

9. CONCLUSION

We have studied the stability of nearly parallel shear flows by means of symmetry methods. In particular, we have presented a complete symmetry classification of the linearized problem in stream-function formulation, showing that in the inviscid case, the algebraic, the exponential, the logarithmic as well as the linear shear base flow admit additional symmetries. In the viscous case, we found additional symmetries for the linear shear flow. The main results of this work can be summarized as follows:

(1) We showed that the Orr-Sommerfeld equation is a symmetry reduction of the linearized Navier-Stokes-Equation using its three basic symmetries.

(2) For a parabolic channel flow, the similarity solution suggests an algebraic grow or decay of the velocity-amplitude of the perturbations, depending on a symmetry parameter β.

(3) Taking a logarithmic base flow, we showed that the invariant solution with respect to the full set of symmetries suggests a linear increase of the wavelength of periodic perturbations.

(4) An exponential boundary layer flow admits symmetries for which the similarity solution suggests a movement of perturbations away from the wall.

(5) In the case of a linear shear flow, we have shown that the generally known solution of the viscous problem is a symmetry reduction using the whole set of symmetries except the translational invariance in time. Including the symmetry for the translational invariance in t leads to four independent invariant basis solutions. Using the same set of symmetries in the inviscid case yields a closed formulation of a similarity solution, which exhibits the behavior of stable modes.

Concluding, we have shown for the famous normal mode approach and for the known solution of the viscous problem for a linear shear flow, that these approaches are symmetry reductions using different sets of symmetries. This surprising result gives support to the approach of solving problems in stability theory by means of symmetry methods. Particularly, it reinforces the results obtained in this work for several special base flows.

In our further research, the influence of the symmetry parameter β onto the similarity solution of the parabolic base flow will be studied. Furthermore, the parameters obtained in the solution for the case of a linear shear flow remain to be interpreted physically and compared to experiments or other analytical approaches. Finally, the influence of boundary conditions on linear combination of similarity solutions remains to be addressed.

10. ACKNOWLEDGEMENT

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[1] **Bluman, G.W und Kumei, S..** *Symmetries and Differential Equations.* Berlin : Springer, 1989

[2] **Steeb, W.-H.** *Continuous Symmetries, Lie Algebras, Differential Equations and Computer Algebra.* 2. s.l. : World Scientific, 2007

[3] **Frewer, M.** *Symmetrie und Selbstähnlichkeit in der Strömungsmechanik.* unpublished, 2006

[4] **Cheviakov, A.F.***Gem software package for computation of symmetries and conservation laws of differential equations.* Computer Physics Communications, 176:48-61, 2007

[5] **Vu, K. and Carminati, J.** *Symbolic Computation and differential equations: Lie Symmetries* J. Symbolic Computation, 29:95-116, 2000

[6] **Moffatt, H.K.** *The interaction of turbulence with strong wind shear.* Pages 139-156. Moscow. Nauka, 1967

[7] **Nold, A.** *Studienarbeit: Symmetry Analysis in Hydrodynamic Stability Theory.* TU-Darmstadt, 2009