MACH NUMBER, RELATIVE THICKNESS, SWEEP AND LIFT COEFFICIENT OF THE WING – AN EMPIRICAL INVESTIGATION OF PARAMETERS AND EQUATIONS

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ABSTRACT

12 equations were investigated to calculate the relative thickness *t/c* of the wing of an aircraft. The calculated relative thickness was taken as the average relative thickness of the wing. The data obtained from these 12 equations was checked against the given average relative thickness of 29 carefully selected aircraft, spanning a space of the parameters Mach number, lift coefficient, sweep, and type of airfoil. Some equations selected are empirical in their nature (partly based on aerodynamic derivation) other equations are purely statistical. Whenever equations had free parameters, these were optimized against the aircraft data. The best equation turned out to be an equation based on nonlinear regression. It achieved a Standard Error of Estimate of only 0.75 % for the average relative thickness of the wing. TORENBEEK's equation will probably be preferred by those that like to see an equation that is based on aerodynamic considerations. It achieved a Standard Error of Estimate of 0.80 % when all its free parameters were considered for optimization. The worst equation produced an Standard Error of Estimate of 8 %. For an airfoil with 10 % relative thickness this would give an unacceptable 10 % ± 8 % mean band of values for *t/c*. This shows that it is necessary to carefully select a suitable equation among the many equations and methods available. Some equations seem not to provide useful results.

1. INTRODUCTION

1.1. Motivation

An accurate sizing of the wing has a significant importance. The wing area depends on lift requirements, mainly during takeoff and landing, as well as on the required fuel volume. Buffet-free high altitude flight may require a somewhat larger wing area. The application of an efficient but complex high lift system can help to reduce wing area requirements. The cruise lift coefficient follows from wing area, aircraft weight, speed. and cruise altitude. Sweep is required to achieve high cruise Mach numbers at tolerable wave drag, however at reduced maximum lift coefficient and further aerodynamic disadvantages. Low relative thickness similarly helps to achieve high Mach numbers, however with detrimental effects on wing stiffness, wing weight and fuel volume. Larger *t/c* tends to increase maximum lift coefficient up to a point, depending on the high lift system, but gains above about 12% relative thickness are small. All of this is influenced by the type of airfoil that is more or less suitable for transonic flow conditions.

Without going into details at this point, it becomes clear that dependencies among the parameters mentioned are plenty and not easy to put into simple equations. This paper limits its view to some key parameters: Mach number, relative thickness, sweep, and lift coefficient of the wing. In addition, the type of airfoil is considered.

A common aircraft design point of view looking at these parameters may be this: For a required cruise Mach number a suitable sweep angle is selected from statistics. The lift coefficient in cruise follows from aircraft weight, wing area and altitude. Relative thickness *t/c* is now calculated from these parameters based on a selected type of airfoil and a chosen level of wave drag. It is this equation for the calculation of *t/c* that is the topic of this paper.

The aim of this paper is to present the results of a literature review with further development of equations that relate the parameters Mach number, relative thickness, sweep, and lift coefficient to one another. Equations will be presented in a form that yields relative thickness as the result.

1.2. Literature

There are a number of equations presented in the literature trying to establish a relationship among the parameters that are of interest in this paper. The different equations are in detail presented in Chapter 3. In combination with the List of References, the origin of the equations can be found. No reference has been found in the literature that

a) extensively compares these equations with one another or
b) tries to check the equations against a large set of statistical data.

This paper is based on a report by CIORNEI [1] that gives more detailed account of the work.

2. FUNDAMENTALS

Mach number, *M*

"The ratio of the true airspeed to the speed of sound under prevailing atmospheric conditions." [2]

Free stream Mach number

The free stream Mach number is the Mach number of the moving body $M = M_\infty = v/a$ with *v* being the true airspeed and *a* the speed of sound. [3]

Critical Mach number, M_{cr}

"By definition, that freestream Mach number at which sonic flow is first obtained somewhere on the airfoil surface is called the critical Mach number of the airfoil." [3]

Crest critical Mach number, M_{CC}

"M_{CC} is the freestream Mach number at which the local Mach number, at the airfoil crest, perpendicular to the isobars, is 1.0. The crest is the point on the upper surface of the airfoil tangent to the freestream direction." [4]

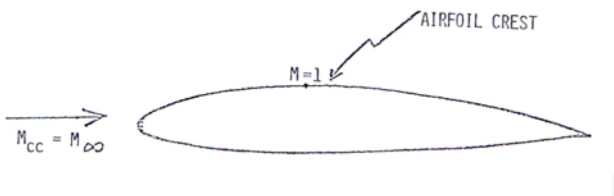

FIG 1. Explanation of M_{CC} [4]

Drag rise Mach number
Drag divergence Mach number, M_{DD}
Drag divergence Mach number, M_{DIV}

"The Mach number beyond which a rapid increase in compressibility drag occurs." [2]

The exact definition of the drag rise Mach number or drag divergence Mach number is arbitrary.

At Airbus and Boeing (compare with [5]) M_{DD} is that Mach number where the wave drag coefficient reaches 20 drag counts ($\Delta C_{D,wave}$ = 0.002).

At Douglas M_{DIV} was defined as that Mach number at which the rate of change in compressibility drag with Mach number is $dC_D / dM = 0.1$ (compare with [5]). SHEVELL defines M_{DIV} as the Mach number at which $dC_D / dM = 0.05$. [4]

The aircraft designer has to decide how much the aircraft should penetrate the flight regime where the aircraft experiences wave drag during cruise flight with M_{CR}. Various philosophies exist. For passenger aircraft the choice is often

(1) $$M_{DD} = M_{CR}$$

This is also what is assumed in this paper for all subsequent calculations.

Effective parameters of swept wings (cosine-rule)

Flow over a swept wing can be divided in a component normal to the quarter cord line V_n and another component tangential to it V_t. The normal flow component may also be called the effective speed V_{eff}. Just based on geometric considerations (cosine-rule), we obtain

(2) $$v_{eff} = v \cos\varphi_{25} \text{ and } M_{eff} = M \cos\varphi_{25} \ .$$

The effective chord normal to the quarter chord line is

(3) $$c_{eff} = c \cos\varphi_{25} \ .$$

The absolute thickness stays the same no matter if the wing is swept or not

(4) $$t_{eff} = t \ .$$

Hence

(5) $$(t/c)_{eff} = (t/c) / \cos\varphi_{25} \ .$$

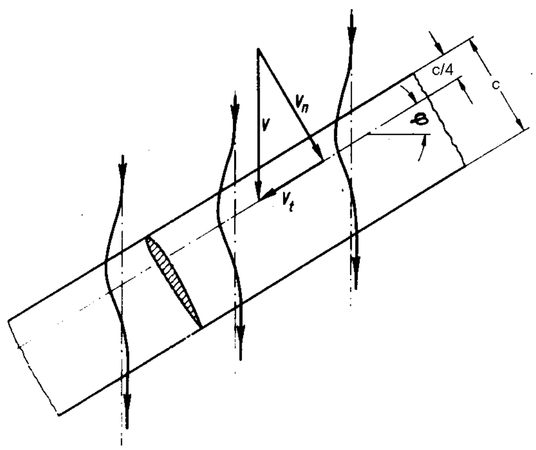

FIG 2. Flow over a swept wing [6]

Effective Mach number (real flows)

The real flow does not necessarily follow the cosine-rule. More generally it can be said that

(6) $$M_{eff} = M \left(\cos\varphi_{25}\right)^x \ .$$

[3] states that 0 < *x* < 1. According to [7]: *x* = 0.5, according to [8]: *x* = 0.75 and by the cosine-rule: *x* = 1.0. Also [9] sets *x* = 1.0. [10] states that *x* is a function of lift coefficient: $x = f(C_L)$.

Effective drag divergence Mach number, $M_{DD,eff}$

We take equation (6) with a value of x = 0.5 as proposed by [7]. Substituting M_{DD} for M into equation (6), we obtain an effective drag-divergence Mach number

$$M_{DD,eff} = M_{DD}\sqrt{\cos\varphi_{25}} \quad . \tag{4}$$

Airfoils in transonic flow

In addition to Mach number, relative thickness, sweep, and lift coefficient of the wing, this report considers also the type of airfoil applied. The more sophisticated the airfoil the higher the Mach number at drag rise.

Conventional airfoils:
"NACA 64-series airfoils ... , although originally designed to encourage laminar flow, turned out to have relative high values of M_{cr} in comparison with other NACA shapes. Hence, the NACA 64 series has seen wide application on high-speed airplanes." [11]

Peaky airfoils:
"The first attempt to modify the general airfoil shape to increase the distance between M_{cr} and M_{DD} was achieved with the invention of the 'peaky' airfoils". [11] "A peaky pressure distribution … intentionally creates supersonic velocities and suction forces close to the leading edge. The airfoil nose is carefully designed so that ... the drag rise is postponed to high speeds." [7]

Supercritical airfoils:
"The purpose of supercritical airfoils is to increase the value of M_{DD}, although M_{cr} may change very little... The supercritical airfoil has a relatively flat top, thus encouraging a region of supersonic flow with lower local values of M than the NACA 64 series. In turn, the terminating shock is weaker, thus creating less drag." [11]

This paper distinguishes arbitrarily between older supercritical airfoils (1965-1987) and modern supercritical airfoils (1988-today).

3. EQUATIONS FOR THE CALCULATION OF THE RELATIVE THICKNESS

Equations from literature could be grouped according to the level of aerodynamic detail included.

a) One extreme are the equations that draw strongly from aerodynamic theory and produce results independent of known aircraft data.
b) On the other extreme are methods purely based on statistical considerations and data regression. These equations are produced independent of aerodynamic derivations.
c) Somewhere in between are equations that show a structure that well represents agreed aerodynamic wisdom but leave free parameters that may help to adjust the equations to known aircraft data.

Categorization of equations can not be performed as strict as it might appear from the above. E.g. for equations of type a) a free parameter may be selected or even be added. In this way the equation can subsequently be considered being of type c).

Not all authors take all influencing parameters into account. Some authors neglect the type of airfoil or the effect of sweep (i.e. only consider unswept wings).

A literature search for equations dealing with the relationship between Mach number, relative thickness, sweep, and lift coefficient of the wing leads to several sources. The equations from these sources are explained in this Chapter.

3.1. Equation based on Torenbeek

TORENBEEK [7] (p. 246) gives an equation in which we see the dependence between the relative thickness and the design Mach number for two-dimensional flow

$$\frac{t}{c} = 0.30\left\{\left[1-\left\{\frac{5+M^2}{5+(M^*)^2}\right\}^{3,5}\right]\frac{\sqrt{1-M^2}}{M^2}\right\}^{2/3} \tag{8}$$

A derivation of this equation is given in [1].

"In this equation M denotes the design (drag-critical) Mach number for which the airfoil is to be designed. The factor M^* ... has no physical meaning and is merely a figure defining the aerodynamic sophistication employed to obtain supercritical flow at the design condition. Good results are obtained by taking:

M^* = 1.0,
conventional airfoils; maximum t/c at about 0.30c
M^* = 1.05,
high-speed (peaky) airfoils, 1960-1970 technology
M^* = 1.12 to 1.15,
supercritical airfoils." [7]

TORENBEEK [7] further writes: "It is difficult to make adequate allowance for the effects of airfoil camber and lift. Provided the airfoil operates at the design c_L, it is possible to use an approximation by reducing M^* ... by .25 times the design c_L for c_L up to .7." and equation (8) becomes:

$$\frac{t}{c} = 0.3\left\{\left[1-\left\{\frac{5+M^2}{5+(M^*-0.25c_l)^2}\right\}^{3,5}\right]\frac{\sqrt{1-M^2}}{M^2}\right\}^{2/3} \tag{9}$$

Equation (9) may be extended to swept wings by including equation (7) and (5). Furthermore the lift coefficient is considered to be that of the wing which is roughly the same as the lift coefficient of the whole aircraft.

$$\frac{t}{c} = 0.3\cos\varphi_{25}$$
$$\left\{\left[1-\left\{\frac{5+M_{DD,eff}{}^2}{5+(M^*-0.25\,C_L)^2}\right\}^{3,5}\right]\frac{\sqrt{1-M_{DD,eff}{}^2}}{M_{DD,eff}{}^2}\right\}^{2/3}$$

(10)

The lift coefficient can also be considered to be an effective value and would need to be modified if the effect of sweep is considered. The expression of this contribution is explain next.

It is $v_{eff} = v\cos\varphi_{25}$ (2), $c_{eff} = c\cos\varphi_{25}$ (3) and $t_{eff} = t$ (4). Lift must support weight (times load factor) no matter if the wing is swept or not. Hence

(11) $$L_{eff} = L \quad .$$

From the definition of lift coefficient

(12) $$L = \frac{1}{2}\varphi\, v^2\, C_L\, S_W$$

or for swept wings

(13) $$L_{eff} = \frac{1}{2}\varphi\, v_{eff}^2\, C_{L,eff}\, S_W \quad .$$

Setting equal these two equations (12) and (13) based on equation (11) results in

(14) $$v_{eff}^2\, C_{L,eff} = v^2\, C_L \quad .$$

Solving for $C_{L,eff}$ yields

(15) $$C_{L,eff} = C_L\,\frac{v^2}{v_{eff}^2}$$

From equation (2) based on geometric considerations

(16) $$\frac{v_{eff}}{v} = \cos\varphi_{25} \quad .$$

But considering aerodynamic effects as explained with equation (6) and applied in (7)

(17) $$\frac{v_{eff}}{v} = \sqrt{\cos\varphi_{25}} \quad .$$

Substituting (17) in (15) yields

(18) $$C_{L,eff} = \frac{C_L}{\cos\varphi_{25}}$$

with C_L from aircraft data and equation (46).

Hence we could also - as a variation to (10) - substitute $C_{L,eff}$ from (18) into (10) and in this way also account for the effect of sweep on lift coefficient.

3.2. Equations from Aerodynamic Similarity based on Anderson

The "transonic similarity equation" by ANDERSON [12] (p. 434)

(19) $$K = \frac{1-M_\infty}{\tau^{2/3}}$$

gives the possibly of a new calculation of the relative thickness τ .

The "transonic similarity equation" states: "Consider two flows at different values of M_∞ (but both transonic) over two bodies with different values of τ, but with M_∞ and τ for both flows such that the transonic similarity parameter *K* is the same for both flows. Then Equation (19) states that the solution for both flows ... will be the same." [12]

The relative thickness τ = *t/c* was determined considering two cases: with or without considering the effect of sweep. Based on Equation (19), the first case was made without considering the effect of sweep. Here M_{DD} was used in place of M_∞ and hence the effect of sweep was not considered

(20) $$t/c = \left(\frac{1-M_{DD}}{K}\right)^{3/2} \quad .$$

In the second case the effect of sweep was considered by using the effective drag divergence Mach number, $M_{DD,eff}$. The equation is this time

(21) $$t/c = \left(\frac{1-M_{DD,eff}}{K_{eff}}\right)^{3/2} \quad .$$

The value of the parameters *K* and K_{eff} were determined based on selected aircraft parameters by using the EXCEL "Solver".

3.3. Equation based on Shevell

Based on SHEVELL [4], the relative thickness *t/c* can be calculated from

(22) $$t/c = f(M_{CC}, \varphi_{25}, C_l)$$

with

(23) $$M_{CC} = \frac{M_{DIV,conventional}}{1.025 + 0.08(1-\cos\varphi_{25})}$$

and

(24) $$M_{DIV,conventional} = M_{DIV,supercritical} - 0.06 \; .$$

Equation (24) was obtained by SHEVELL [4] from a comparison of data related to aircraft with conventional wings and wind tunnel data from a supercritical wing. He concludes that the wave drag curve of a wing with a conventional airfoil is shifted up by 0.06 Mach if a supercritical airfoil is used. For this reason also M_{DIV} is shifted up that amount.

M_{DIV} and M_{DD} are both drag-divergence Mach numbers (see Chapter 2), but following a different definition. Without knowing the shape of the drag rise curve, there is no way to convert M_{DIV} to M_{DD}. From Figure 2 in [4] it can be concluded that $M_{DIV} < M_{DD}$. A *very* crude estimate would be to assume that $M_{DD} = M_{DIV} + 0.02$. If SHEVELL's approach is to be compared with the results of other authors and with aircraft data, this crude assumption should be applied in view of a missing better relation:

(25) $$M_{DIV} = M_{DD} - 0.02$$

Equation (22) is in fact this

$$\frac{M_\infty^2 \cos^2\Lambda}{\sqrt{1-M_\infty^2\cos^2\Lambda}} \cdot \left[\left(\frac{\gamma+1}{2}\right)\frac{2.64(t/c)_\infty}{\cos\Lambda} + \left(\frac{\gamma+1}{2}\right)\frac{2.64(t/c)_\infty(0.34C_L)}{\cos^3\Lambda}\right]$$
$$+\frac{M_\infty^2\cos^2\Lambda}{1-M_\infty^2\cos^2\Lambda} \cdot \left[\left(\frac{\gamma+1}{2}\right)\left[\frac{1.32(t/c)_\infty}{\cos\Lambda}\right]^2\right]$$
$$+M_\infty^2\cos^2\Lambda \cdot \left[1+\left(\frac{\gamma+1}{2}\right)\frac{(0.68C_L)}{\cos^2\Lambda} + \frac{\gamma+1}{2}\left(\frac{0.34C_L}{\cos^2\Lambda}\right)^2\right] - 1 = 0$$

(22)

Here $M_\infty = M_{CC}$, $\Lambda = \varphi_{25}$, and $(t/c)_\infty$ is simply t/c. $\gamma = \frac{C_p}{C_v} = 1.4$

t/c in (22) has to be obtained from an iteration, which is again performed with EXCEL and the modified Newton method of the "Solver".

3.4. Equation based on Kroo

The origin of this equation is based on a graph given by KROO [13] (Figure 7). This graph is based on SHEVELL [14] (p. 199) adapted to swept wings. It is given in this text as FIG 3. KROO explains "This graph displays M_{CC} as a function of the airfoil mean thickness ratio t/c and C_L. It is based on studies of the M_{CC} of various airfoils representing the best state of the art for conventional 'Peaky' type airfoils typical of all existing late model transport aircraft."

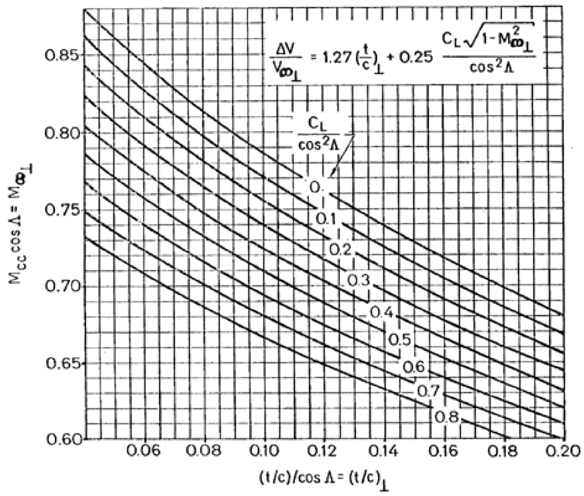

FIG 3. Crest critical Mach number M_{CC} as a function of relative thickness t/c and lift coefficient C_L [13]

For easier handling on a computer, the graph was transformed here into equations. The derivation of this transformation is given in [1].

(26) $$t/c = x\cos\varphi_{25}$$

(27) $$x = \frac{-v-\sqrt{v^2-4uw}}{2u}$$

with

(28) $$u = 2.8355$$

(29) $$v = -1.9072 + 0.2 \cdot 2.131 y$$

(30) $$w = 0.9499 - 0.2y - M_{CC}\cdot\cos\varphi_{25}$$

(31) $$y = C_L / (\cos\varphi_{25})^2$$

(32) $$M_{CC} = \frac{M_{DIV,Peaky}}{1.025 + 0.08(1-\cos\varphi_{25})}$$

where

(33) $$M_{DIV,Peaky} = M_{DIV} + \Delta M_{DIV}$$

M_{DIV} is taken from equation (25). The values of the term

ΔM_{DIV} were obtain using literature information given in TAB 1.

TAB 1. Variation of divergence Mach number ΔM_{DIV} depending on the type of airfoil

type of airfoil	ΔM_{DIV}	source
conventional	+0.04	[7]
peaky	0	[13]
older supercritical	-0.04	[13]
modern supercritical	-0.06	[4], [10], [13]

3.5. Equation from Howe

HOWE [15] (p. 117) offers a relationship between the lift coefficient, the thickness to chord ratio, and the critical Mach number:

(34) $$M_{DD,eff} = A_F - 0.1\, C_L - t/c \quad .$$

"A_F is a number, which depends upon the design standard of the aerofoil section. For older aerofoil A_F was around 0.8 but a value of 0.95 should be possible with an optimized advanced aerofoil." [15]. In effect, we can think of as A_F being the effective drag divergence Mach number of an airfoil of zero thickness at zero lift coefficient. Once the angle of attack is increased and hence the lift coefficient, the drag divergence Mach number will decrease. This is due to the super velocities on the top of the airfoil. According to (34) the drag divergence Mach number also decreases with an increase of relative thickness. The same explanation holds also in this case. The extend to which these phenomena have an influence on the drag divergence Mach number is given in (34) by the factor 0.1 for C_L and the factor 1.0 for t/c. If these factors will in fact result in an optimum representation of measured parameters will be investigated in Chapter 4. Here, the interest is first of all to calculate relative thickness

(34) $$t/c = A_F - 0.1\, C_L - M_{DD,eff} \quad .$$

3.6. Equation from Jenkinson

Using the notation of this paper, JENKINSON's equation [9] (p. 116) reads

$$M_{DD} = 0.9965 - 1.387 \cdot t/c + 4.31 \cdot 10^{-5} \varphi_{25} - 0.18 \cdot C_L$$

(35)

Again we can think of M_{DD} = 0.9965 for a wing with zero relative thickness at zero lift coefficient and with zero sweep. Solving for relative thickness

$$t/c = 0.7185 + 3.107 \cdot 10^{-5} \varphi_{25} - 0.1298 \cdot C_L - 0.7210 \cdot M_{DD}$$

(35)

3.7. Equation from Weisshaar

WEISSHAAR [16] presents an equation to calculate the drag divergence Mach number:

(36) $$M_{DD} = \frac{K_A}{\cos\varphi_{25}} - \frac{t/c}{\cos^2\varphi_{25}} - \frac{C_L}{10\cos^3\varphi_{25}}$$

where K_A is approximately 0.80 ... 0.90.

We can think of K_A as being the drag divergence Mach number of an unswept wing of zero thickness at zero lift coefficient.

M_{DD} increases with sweep by a factor of 1 / cos φ_{25}. The relative thickness t/c can then be calculated from

(36) $$t/c = K_A \cos\varphi_{25} - M_{DD}\cos^2\varphi_{25} - \frac{C_L}{10\cos\varphi_{25}} \quad .$$

3.8. Equation based on Böttger

BÖTTGER [17] presents three graphs. These three graphs can be used to determine the drag divergence Mach number M_{DD} (called M_{20} by [17]). The aim here was to convert the graphs to an equation in order to use the information on a computer for an easy comparison with the other methods. The transformation of the graphs into equations is shown in [1]. The results are

(37) $$\begin{aligned} M_{DD} &= a(C_L - b)^d + c \\ &- 30/27 \cdot (t/c - 0.113) \\ &+ 0.00288 \cdot (\varphi_{25} - 29.8°) \end{aligned}$$

with
a = -1.147
b = 0.200
c = 0.838
d = 4.057

Solved for the relative thickness the equation becomes

$$t/c = \frac{27}{30}\left[a(C_L - b)^d + c + 0.00288(\varphi_{25} - 29.8°) - M_{DD}\right] + 0.113$$

(37)

3.9. Equation based on Raymer

RAYMER [5] (p. 293) gives "a preliminary estimate of wing M_{DD}"

(38) $$M_{DD} = M_{DD}(C_L = 0)\ LF_{DD} - 0.05 \cdot C_L$$

Two diagrams are used in [5] to obtain $M_{DD}(C_L = 0)$ and LF_{DD}. Here equations are given instead. The derivation is shown again in [1].

$$M_{DD}(C_L = 0) =$$
$$1 + k_{M,DD}\left(u(90° - \rho_{25})^3 + v(90° - \rho_{25})^2 + w(90° - \rho_{25})\right)$$

(39)

with u = 8.029·10^{-7} 1/deg^3
v = -1.126·10^{-4} 1/deg^2
w = 8.437·10^{-4} 1/deg

(40) $$k_{M,DD} = 1317 \cdot (t/c)^3 - 324.3 \cdot (t/c)^2 + 28.948 \cdot (t/c) - 0.0782$$

(41) $$LF_{DD} = k_{LF,DD}\left(a\ C_L^{\ 2} + b\ C_L\right) + 1$$

with a = -0.1953
b = -0.1494

(42) $$k_{LF,DD} = 23.056 \cdot (t/c)^2 + 3.889 \cdot (t/c)$$

In order to compare RAYMER's equation with that of the other authors, it is handy to calculate the relative thickness. This obviously is not that easy, since *t/c* is included in several equations. The relative thickness can be calculated iteratively. A modified Newton method given in EXCEL by means of the "Solver" has been used here to calculate for *t/c*.

3.10. Equation based on Linear Regression

In linear regression, the model is built as a linear combination of the independent variables. In the case treated here, a multiple regression model could be set up like this

(43) $$t/c = a\ M_{DD} + b\ \varphi_{25} + c\ C_L + d\ k_m \quad .$$

This is a 5-dimensional data modeling exercise, because 4 independent plus one dependent variable are involved. k_m is a parameter accounting for the sophistication of the airfoil in the same way as M* does this in TORENBEEK's equation. The task in multiple regression is to find a set of parameters *a*, *b*, *c*, and *d* that best fit the given data.

Here, we follow a slightly modified approach to linear regression. Instead of using M_{DD} and φ_{25} separately as in (43), we use the nonlinear combination of the two parameters

(7) $$M_{DD,eff} = M_{DD}\ \sqrt{\cos\varphi_{25}}$$

following aerodynamically proven evidence. This than yields a modified regression model

(44) $$t/c = a\ M_{DD,eff} + b\ C_L + c\ k_m \quad .$$

The parameters are fit to the data in this project by help of the EXCEL „Solver".

3.11. Equation based on Nonlinear Regression

In nonlinear regression the regression equation is of nonlinear form. Virtually any function know from mathematics could be used for curve fitting. The technical understanding about the particular problem should lead to the selection of a suitable equation. Popular are among others Taylor Series Equations, Polynomial Equations, or Power Family Equations – just to name a few. Here, a standard approach is followed, using a variant of the Power Family Equations

(45) $$t/c = k_t \cdot M_{DD}^{\,t} \cdot \cos\varphi_{25}^{\,u} \cdot c_L^{\,v} \cdot k_M^{\,w} \quad .$$

It was decided to take cos φ_{25} as one of the independent parameters instead of φ_{25}, in order to resemble the flow phenomena better. As in all the other cases, the parameters are fit to the data with the EXCEL "Solver".

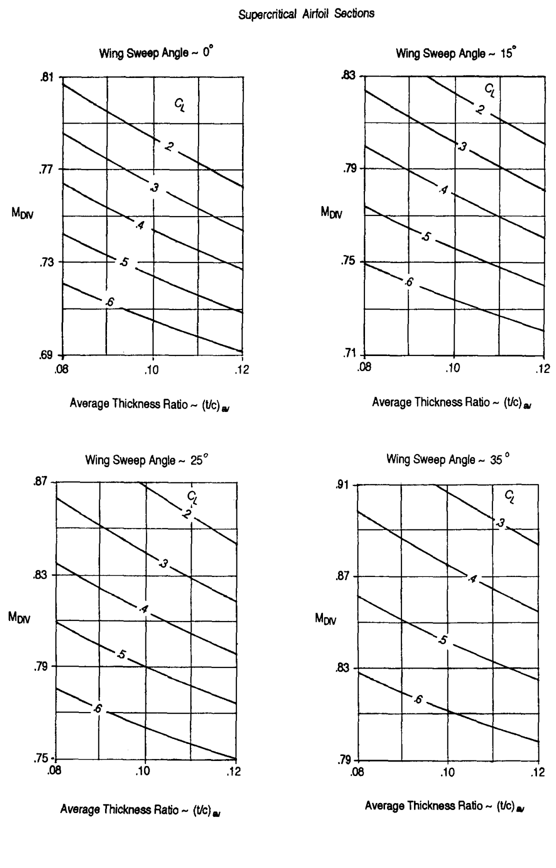

FIG 4. Determination of M_{DIV} or *t/c* from [10]

3.12. Diagram from Schaufele

The parameters of interest are put in relation by a diagram from SCHAUFELE [10] (p. 102). The diagram is based on supercritical airfoils. No equations have been determined from this type of graphical data. Instead, the graphs have been evaluated manually.

4. INVESTIGATION, COMPARISON, AND ADAPTATION OF EQUATIONS

A complete evaluation of all 12 equations presented in Chapter 3 is the next step in the investigation. The idea is to check the equations given and to optimize free parameters based on data of 29 carefully selected aircraft.

4.1. Input from Aircraft Data

With the idea that free parameters of equations shall be fitted to aircraft data, it is evident that aircraft had to be selected carefully. A selected inadequate set of aircraft (and aircraft data) could easily lead to wrong results when fitting (optimizing) parameters. So the aim was to select a set of aircraft that

- span well the parameter range in question,
- represent well the history of aerodynamic evolution.

The aircraft chosen cover a range of different values of sweep (from 0° to 35°), different drag divergence Mach numbers (from 0.65 to 0.88), different average relative wing thickness (from 9% to 13.4%), cruise lift coefficient (from 0.22 to 0.73), and type of airfoil (conventional, peaky, older supercritical airfoils, and modern supercritical airfoils). Every parameter was taken from more then two sources of documentation, so that there is some assurance of the accuracy of the aircraft data. The aircraft are presented with their three-view drawing and with the reviewed data from several sources in [1].

The selected aircraft are grouped according to their airfoil class: conventional airfoil, peaky airfoil, older supercritical airfoil, and modern supercritical airfoil.

Selected aircraft with conventional airfoil(s):

- IAI 1124A Westwind 2
- Sud Aviation Caravelle
- VFW 614
- HFB 320
- Gates Lear Jet Model 23
- Lockheed C-141 Starlifter
- Lockheed Jetstar II
- Dassault Falcon 20

Selected aircraft with peaky airfoil(s):

- BAC One-Eleven Series 500
- McDonnell Douglas DC-9 Series 30
- Vickers VC-10 Super VC-10
- McDonnell Douglas DC-8 Series 63
- McDonnell Douglas DC-10 Series 10
- Lockheed C-5A

Selected aircraft with older supercritical airfoil(s):

- Mitsubishi Diamond I
- Airbus A300-600
- Boeing 767-200
- Cessna 650 Citation VI
- Airbus A310-300
- Raytheon Hawker 800XP
- Raytheon Beechjet 400A
- Beriev Be-40

Selected aircraft with modern supercritical airfoil(s):

- Bombardier Global Express
- Bombardier Challenger CRJ 200 LR
- Tupolev Tu-204-300
- BAe RJ85
- Embraer EMB-145
- Airbus A321-200
- Airbus A340-300

As pointed out in Chapter 3, the equations under investigation relate Mach number, relative thickness, sweep and lift coefficient of the wing to one another. It was decided to consider the relative thickness *t/c* as the unknown and the other parameters as known inputs. But also these input parameters had to be evaluated first. The ideas behind these calculations are discussed next:

The **Mach number** to enter calculations of the relative thickness is the drag divergence Mach number M_{DD} – that Mach number at which the aircraft would experience 20 drag counts. In other words:

- M_{DD} was taken as M_{MO} (following Boeing and Airbus design principles) if M_{MO} was known,
- M_{DD} was taken as a Mach number (calculated from V_{MO} and a known or assumed altitude h up to which V_{MO} is flown) if M_{MO} was unknown or considered to be unrealistic.

The **lift coefficient** C_L to enter further calculations was calculated from

- the mass in cruise flight m_{CR}. m_{CR} was assumed to be equal to the maximum take-off mass m_{MTO},
- the cruise speed V_{CR} calculated from M_{DD} in altitude h as given above,
- the density ρ in altitude h,
- the reference wing area S_W.

(46) $$C_L = \frac{2\, m_{MTO}\, g}{\rho\, V_{CR}^{\;2}\, S_W}$$

The **average relative thickness of the wing** *t/c* was calculated from wing tip and wing root relative thickness with an equation from JENKINSON [9]

(47) $$t/c = \frac{3\,(t/c)_{tip} + (t/c)_{root}}{4} \quad .$$

In some cases where an average relative thickness of the wing was given in the literature this value was taken for further calculations.

The **wing sweep at 25% chord** φ_{25} was given or could easily be determined.

4.2. Calculation, Optimization, and Results

The equations can be split in two parts (see TAB. 4):

- equations with fixed parameters
- equations with parameters that are free for optimization.

Equations with fixed parameters are equations which are ready to calculate relative thickness. All factors and parameters are given. **Equations with free parameters** are equations that include parameters either unknown or free for adaption. These parameters may be fitted to given aircraft data. In this way the output value for *t/c* may be optimized.

In any of these two cases the result of the calculation is the **Standard Error of Estimate** *SEE*. This value tells us how far off our estimate of the relative thickness is, when compared with actual aircraft data.

(48) $$SEE = \sqrt{\frac{\sum \left(y_{estimate} - y\right)^2}{n}}$$

In this equation $y_{estimate}$ is the value (here the relative thickness, *t/c*) that was calculated, *y* is the given value from the aircraft, *n* is the number of test calculations (here *n* = 29).

For each aircraft and each equation we get an *error* ² that is $(y_{estimate} - y)^2$. Summing up all the *error* ² calculated with one equation for all *n* = 29 aircraft should be as low as possible.

$$\frac{\sum \left(y_{estimate} - y\right)^2}{n}$$

is the average *error* ². Taking the square root of *error* ² yields the average error know as the Standard Error of Estimate (*SEE*). Note that the *SEE* shows an absolute error. In case of the relative thickness we deal with relative values (in %). Nevertheless the *SEE* is absolute with respect to the results of *t/c*. This can be made clear using an example. An aircraft has a relative thickness of 10% the *SEE* was calculated to be 1%. This means that on average we expect *t/c* values from our equation that are off by an absolute 1%, i.e. we may expect results like *t/c* = 9% or *t/c* = 11%.

The **optimization** of the equation means to determine optimized values of the free parameters. This leads to a minimum Standard Error of Estimate. Thus the results obtained are the best results possible with the equation in question and selected aircraft data. They are quite close to the real values of the relative thickness of the aircraft. This best fit was calculated with EXCEL and the modified Newton method of the “Solver”. The “Solver” drives the *SEE* to a minimum.

Torenbeek's equation can be considered an equation with fixed parameters. Nevertheless all its parameters have been questioned and opened up for optimization. Two cases can be further distinguished:

- with consideration of sweep in the calculation of C_L, hence using $C_{L,eff}$ from equation (18) in (10),
- without the contribution of sweep, hence using C_L in (10).

It turned out that these two variants produce only small differences in the results. The version taking the lift coefficient C_L straight into the equation without considering sweep effects on lift produced slightly better results.

The fixed parameters in TORENBEEK's equation 0.3 and 2/3 were opened for optimization. Equation (10) now reads:

$$\frac{t}{c} = k_T \cos\varphi_{25}$$
$$\left\{\left[1 - \left\{\frac{5 + M_{DD,eff}{}^2}{5 + (M^* - 0.25\, C_L)^2}\right\}^{3,5}\right] \frac{\sqrt{1 - M_{DD,eff}{}^2}}{M_{DD,eff}{}^2}\right\}^E$$

TORENBEEK's equation with its parameters in standard form (as proposed by [7]) produced a *SEE* of 2.88 %. With all parameters free for optimization EXCEL calculated a *SEE* of only 0.80 %. Parameters are shown in TAB 2.

TAB 2. Parameters used for TORENBEEK's equation

parameter	standard	optimized
M* for conventional	1.000	0.907
M* for peaky	1.050	1.209
M* for older supercritical	1.135	4.703
M* for modern supercritical	1.135	1.735
k_T	0.300	0.130
E	0.667	0.038

One problem with opening up parameters for optimization is that parameters are driven to values that do not have physical meaning in the end. If *M** can be seen as the local maximum Mach number on the surface (of an unswept wing) when the aircraft flies with a speed of M_{DD} then a value of *M** = 4.7 for a supercritical wing does not make much sense. On the other hand we need to except parameters without physical meaning if we want to benefit from an optimized fit of parameters to aircraft data and best results from the calculation.

The results for **Howe's equation** are given in TAB 3. The optimized values agree well with those given by Howe. However, the peaky airfoils seem to be better than they are known to be.

TAB 3. Parameter A_F used for HOWE's equation

A_F	standard	optimized
A_F for conventional	0.80	0.861
A_F for peaky	0.85	0.935
A_F for older supercritical	0.90	0.907
A_F for modern supercritical	0.95	0.926

For the calculation of *t/c* **aerodynamic similarity based on Anderson** without considering sweep, optimization found *K* = 1.714. For the calculation of *t/c* from similarity with considering sweep, optimization found: K_{eff} = 1.890.

For **linear regression** the optimization found

- a = 0.1460
- b = -0.00513
- c = 0.00257

For **nonlinear regression** the optimization found

- k_T = 0.127
- t = -0.204
- u = 0.573
- v = 0.065
- w = 0.556

and parameters k_M from TAB 4.

TAB 4. Parameters k_M used for equation (45) from nonlinear regression

parameter	value
k_M for conventional	0.921
k_M for peaky	0.928
k_M for older supercritical	1.017
k_M for modern supercritical	0.932

For **Weisshaar's equation** optimization yields K_A = 0.887. This is in the range for this parameter as given by WEISSHAAR who wrote: K_A is approximately 0.80 ... 0.90.

The accuracy with which the relative thickness was calculated with the equations investigated, is given in TAB 5.

TAB 5. Comparison of different equations used to calculate the relative thickness of a wing based on the Standard Errors of Estimate

ranking	method	SEE	optimized
1	nonlinear regression	0.75 %	yes
2	TORENBEEK (with term C_L)	0.80 %	yes
3	linear regression	1.18 %	yes
4	similarity with sweep	2.43 %	yes
5	HOWE	3.67 %	yes
6	similarity without sweep	3.71 %	yes
7	WEISSHAAR	3.95 %	yes
8	JENKINSON	4.23 %	no
9	BÖTTGER	4.32 %	no
10	RAYMER	4.54 %	no
11	KROO	4.59 %	no
12	SHEVELL	8.06 %	no
	average SEE	3.25 %	

SCHAUFELE's diagrams produced a SEE of about 3.3 %.

5. CONCLUSIONS

The aim of this project was to search and develop equations that relate the parameters Mach number, relative thickness, sweep, and lift coefficient to one another. 12 equation were found in the literature. The equations were taken from divers sources. Some equations draw strongly from aerodynamic theory but other equations are purely based on statistical considerations and data regression. In a few cases the starting point in the determination of the equations where diagrams that first needed to be converted into equations.

For the calculations done with these 12 equations, 29 transport aircraft were used. The aircraft chosen cover a wide range of paramters and types of airfoil.

The accuracy of the equations found in literature was improved by adaptation / optimization of the free parameters with respect to the data base of 29 aircraft. For those equations with fixed parameters just the accuracy was calculated.

The best results were – as expected – achieved by the optimized methods. The equation based on nonlinear regression can be recommended. TORENBEEK's equation will probably be preferred by those who like to see an equation that is based on aerodynamic considerations.

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